电动力学讲稿（一）

2022.3.15

关于矢量变换规则的讨论

坐标系变换 → 矢量的分量按照某种方式变换.

目前这个阶段:

When we rotate the coordinate system, the components of the vector transform among themselves in the correct way.

①

$S = \vec{A} \cdot \vec{B}$

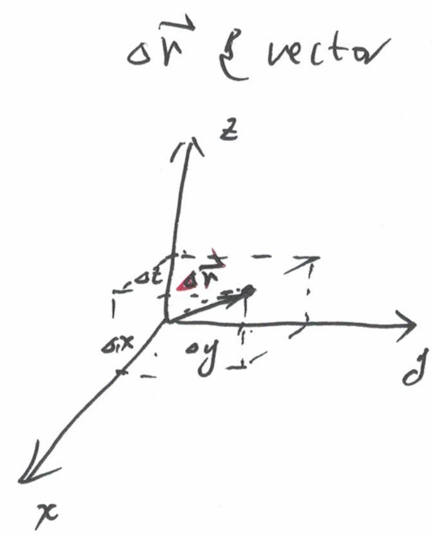
是否找到一个标量, 能写成两个矢量的乘积

$T_1 = T(x, y, z)$

$T_2 = T(x+\Delta x, y+\Delta y, z+\Delta z)$

$\Rightarrow \Delta T = T_2 - T_1$

$= \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y + \frac{\partial T}{\partial z}\Delta z$



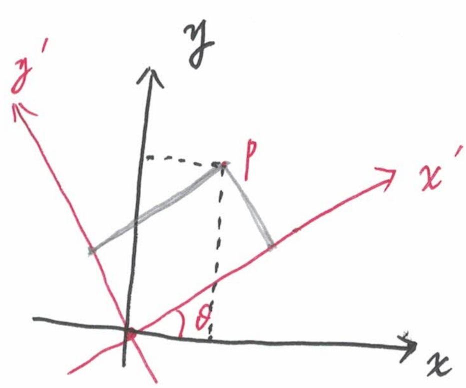
$(\Delta x, \Delta y, \Delta z)$ 是一个 vector $= \Delta\vec{r}$

则 $\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right)$ 也是一个 vector

$= \nabla T \cdot \Delta\vec{r}$

Not convinced ?

我们做一个坐标系变换.



$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

$$\Longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\Delta x^{0} = \Delta x'\cos\theta - \Delta y'\sin\theta$$
$$\Delta y^{0} = \Delta x'\sin\theta + \Delta y'\cos\theta$$

$$\Delta T = \frac{\partial T}{\partial x'}\Delta x' + \frac{\partial T}{\partial y'}\Delta y'$$
$$= \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y$$
$$= \frac{\partial T}{\partial x}\left(\Delta x'\cos\theta - \Delta y'\sin\theta\right) + \frac{\partial T}{\partial y}\left(\Delta x'\sin\theta + \Delta y'\cos\theta\right)$$
$$= \left(\frac{\partial T}{\partial x}\cos\theta + \frac{\partial T}{\partial y}\sin\theta\right)\Delta x' + \left(-\frac{\partial T}{\partial x}\sin\theta + \frac{\partial T}{\partial y}\cos\theta\right)\Delta y'$$

$$\Longrightarrow \begin{cases} \dfrac{\partial T}{\partial x'} = \dfrac{\partial T}{\partial x}\cos\theta + \dfrac{\partial T}{\partial y}\sin\theta \\[2ex] \dfrac{\partial T}{\partial y'} = -\dfrac{\partial T}{\partial x}\sin\theta + \dfrac{\partial T}{\partial y}\cos\theta \end{cases}$$

可见 $\left(\frac{\partial T}{\partial x'}, \frac{\partial T}{\partial y'}\right)$ 与 (x', y') 的变换是一致的！

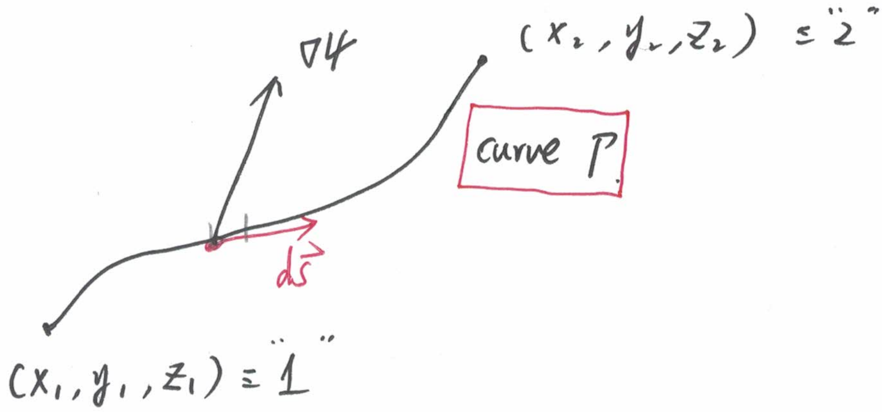
为矢量 $\Longrightarrow$ ∇T 为梯度

矢量场：如何分析

我们手上已经有了一个矢量场了！ $\nabla\psi$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{x} + \frac{\partial\psi}{\partial y}\hat{y} + \frac{\partial\psi}{\partial z}\hat{z}$$

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ 也是一个矢量.



$$\int_{(1),\ \text{curve }\Gamma}^{(2)} \nabla\psi\cdot\vec{ds} = ?$$

↘ 积分值是否依赖于 curve？

$\psi(x+\Delta x, y+\Delta y, z+\Delta z) - \psi(x,y,z)$

↘ 这些点均在线上.

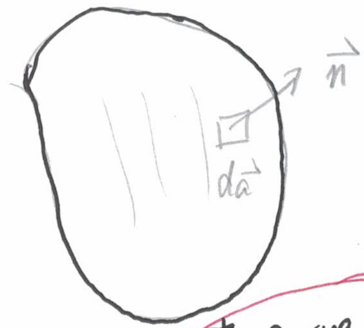
$= \nabla\psi\cdot\Delta\vec{s}$

$= \psi(2) - \psi(1)$

与 curve Γ 无关！（但是 curve 必须光滑！）

在每个点上都有一个 $\nabla 4$，我们在某一个曲面上积一下分.

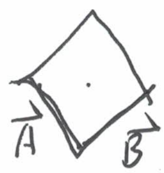
$$\int_S \nabla 4 \cdot d\vec{a}$$



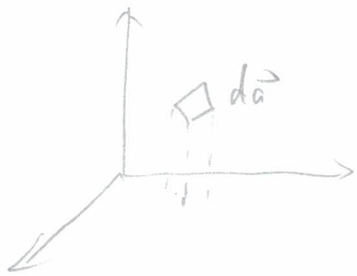
$d\vec{a} = da \cdot \vec{n}$

由 curve 了包围的一个曲面

what is $da\,\vec{n}$ ~~$=dx\,d$~~



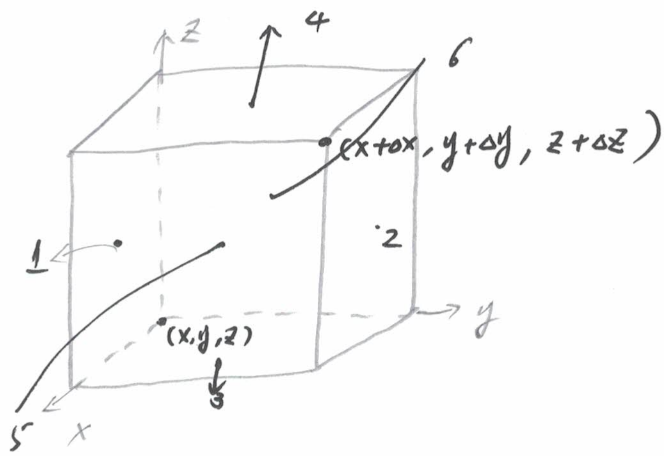
$\vec{A} \times \vec{B} = AB\sin\theta \cdot \vec{n}$



进行投影!

$$\int_S \nabla 4 \cdot \vec{n}\, da$$

一个特殊的曲面:



$$\int_S \vec{A} \cdot \vec{n}\, da$$

$$= -A_y(1)\,\Delta x\Delta z + A_y(2)\,\Delta x\Delta z$$

$$+ A_x(5)\,\Delta y\Delta z - A_x(6)\,\Delta y\Delta z + A_z(4)\,\Delta x\Delta y - A_z(3)\,\Delta x\Delta y$$

$$= \frac{\partial A_y}{\partial y}\Delta y \Delta x \Delta z + \frac{\partial A_x}{\partial x}\Delta x \Delta y \Delta z + \frac{\partial A_z}{\partial z}\Delta z \Delta x \Delta y$$

$$= \nabla\cdot\vec{A}\,\Delta x \Delta y \Delta z$$

$$\boxed{\int_S \vec{A}\cdot\vec{n}\,da = \int \nabla\cdot\vec{A}\,dV}$$ Gauss 定理.

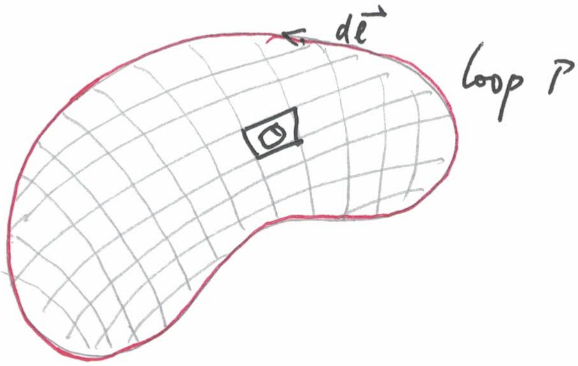
what is $\nabla\cdot(\nabla\psi) \equiv \nabla^2\psi$

what is the meaning of $\nabla\cdot\vec{A}$?

$$\int_{\square} \vec{A}\cdot\vec{n}\,ds = \nabla\cdot\vec{A}\,\Delta V$$

$$\Longrightarrow \nabla\cdot\vec{A} = \left.\frac{\int_{\square}\vec{A}\cdot\vec{n}}{\Delta V}\right|_{\text{at a point}}$$

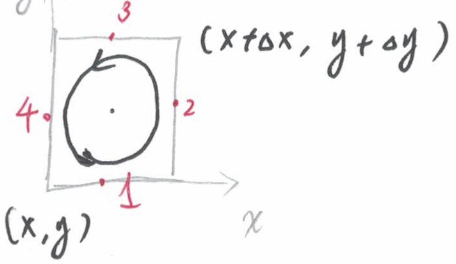
矢量 $\vec{A}$ 还可以沿着闭合路径积分.



$$\int \nabla\psi\cdot d\vec{l} = 0$$ ，显然

曲线闭合包围曲面. → ?

曲面闭合包围体积 → Gauss 定理



$$\int \vec{A}\cdot d\vec{l} = A_x(1)\,\Delta x + A_y(2)\,\Delta y - A_x(3)\,\Delta x - A_y(4)\,\Delta y$$

$$= \big(A_x(1) - A_x(3)\big)\Delta x + \big(A_y(2) - A_y(4)\big)\Delta y$$

$A_x(1) \downarrow A_x(x,y)$ $\quad A_x(3) \downarrow A_x(x, y+\Delta y)$ $\quad A_y(2) \downarrow A_y(x+\Delta x, y)$ $\quad A_y(4) \searrow A_y(x,y)$

$$= -\frac{\partial A_x}{\partial y}\Delta y\,\Delta x + \frac{\partial A_y}{\partial x}\Delta x\,\Delta y$$

$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\Delta x\,\Delta y$$

$$= (\nabla\times\vec{A})_z \cdot d\vec{a}$$

$$\oint \vec{A}\cdot d\vec{l} = (\nabla\times\vec{A})_n\,\Delta a$$

$\oint \vec{A}\cdot d\vec{l}$ ↓ 环量

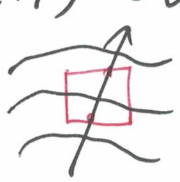
n ↘ which n

$$\rightsquigarrow (\nabla\times\vec{A})_n = \frac{\oint_l \vec{A}\cdot d\vec{l}}{\Delta a}$$ 这就是物理含义

为什么 $\nabla\psi$ 无环量

①、数学上 $\nabla\times(\nabla\psi) = 0$

②.



③

Summary:

Q: ~~如何~~ 上面的分析似乎有些糊子摸了。我们想知道一个矢量场的性质 ⇝ 要分析其空间变化 ⇝ 我们分析了散度和旋度。(知道了一个矢量场的散度和旋度，还能能知道矢量场 $\vec{A}(x,y,z)$，先不考虑时间问题)。

↳ 动力学

$$\nabla \cdot \vec{A} = f_0(x,y,z)$$

$$\nabla \times \vec{A} = f_1(x,y,z)\,\hat{x} + f_2(x,y,z)\,\hat{y} + f_3(x,y,z)\,\hat{z}$$

⇝ 未知函数 $A_x(x,y,z),\ A_y(x,y,z),\ A_z(x,y,z)$

4个一阶偏微分方程

$$\begin{cases} \dfrac{\partial A_x(x,y,z)}{\partial x} + \dfrac{\partial A_y(x,y,z)}{\partial y} + \dfrac{\partial A_z(x,y,z)}{\partial z} = f_0(x,y,z) \\ \dfrac{\partial A_z(x,y,z)}{\partial y} - \dfrac{\partial A_y(x,y,z)}{\partial z} = f_1(x,y,z) \\ \dfrac{\partial A_x(x,y,z)}{\partial z} - \dfrac{\partial A_z(x,y,z)}{\partial x} = f_2(x,y,z) \\ \dfrac{\partial A_y(x,y,z)}{\partial x} - \dfrac{\partial A_x(x,y,z)}{\partial y} = f_3(x,y,z) \end{cases}$$

假定知道边界条件如
$\vec{A}(x_0, y_0, z_0)$ 知道
$\vec{A}$ 在某一点的导数知道

$$\vec{A} \equiv \vec{A}_d + \vec{A}_c$$

$\vec{A}_c$ → curl free

$\vec{A}_d$ → divergent free

$$\nabla \cdot \vec{A}_d = 0, \quad \nabla \times \vec{A}_c = 0.$$

⟹ 亥姆霍兹定理.

（若矢量场 $\vec{A}$ 在无界空间中处处单值，且其导数连续有界，则该矢量场唯一地由其散度和旋度所确定）

⟹ $\nabla \cdot \vec{A}_c = f_0(x,y,z)$

$$\nabla \times \vec{A}_d = f_1(\vec{r})\hat{x} + f_2(\vec{r})\hat{y} + f_3(\vec{r})\hat{z}$$

然后知道 $\vec{A}_c(\vec{r}) + \vec{A}_d(\vec{r})$ 在某点边界的行为

假定是无旋场，只有 $\vec{A}_c$

$\nabla \cdot \vec{A}_c = f_0(x,y,z)$，知道 $\vec{A}_c$ 在边界上的行为，是否能求出 ~~电场~~ $\vec{A}_c$.

⟿ 静电场

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \vec{E} = 0$$

⟹ $\vec{E} = -\nabla\phi$

⟹ $\nabla^2\phi = -\frac{\rho}{\varepsilon_0}$ ⟿ ~~Laplace~~ 泊松 方程.

知道 ρ 的分布

⇝ 假定无散度场

$$\nabla\cdot\vec{A}_d(\vec{r}) = 0$$

$$\nabla\times\vec{A}_d(\vec{r}) = f_1(\vec{r})\,\hat{x} + f_2(\vec{r})\,\hat{x} + f_3(\vec{r})\,\hat{y}$$

$$\vec{A}_d = \nabla\times\vec{C}$$

$$\nabla\times(\nabla\times\vec{C}) = \nabla(\nabla\cdot\vec{C}) - \nabla^2\vec{C} = \cdots$$

三个方程，三个函数，某种意义上有解.

介电体静电学/静磁学

Maxwell 方程

$$\nabla \cdot \vec{E} = \frac{\rho_t}{\varepsilon_0}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}_t + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

+ 边界条件

ρ_t: total charge density (free charge density; polarization charge density)

$\vec{j}_t$: total current density (free ~~cha~~ current density; current density due to charge polarization; current density due to magnetization)

a) Free charge density ρ_f

b) charge density due to polarization

$$\rho_p = -\nabla \cdot \vec{P}$$

$\vec{P}$ 为电极化强度矢量，或者说电偶极矩密度

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a) free current density $\vec{j}_f$

b) current density due to charge polarization

$$\vec{j}_p = \frac{\partial \vec{P}}{\partial t}$$

c). the current density due to magnetization

$$\vec{j}_M = \nabla \times \vec{M}$$

$\vec{P}$ 与 $\vec{M}$ 是 材料属性.（各种各样机制）

$$\nabla \cdot \vec{E} = \frac{\rho_f - \nabla \cdot \vec{P}}{\varepsilon_0}$$

$$\rightsquigarrow \nabla \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \frac{\rho_f}{\varepsilon_0}$$

$\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}$ ，电位移矢量

$$\nabla \times \vec{B} = \mu_0 \left(\vec{j}_f + \frac{\partial \vec{P}}{\partial t} + \nabla \times \vec{M} \right) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\rightsquigarrow \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{j}_f + \frac{\partial \vec{D}}{\partial t}$$

$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ ，磁场强度矢量

⇝ 介质中 Maxwell 方程组：

$$\nabla \cdot \vec{D} = \rho_f$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{j}_f + \frac{\partial \vec{D}}{\partial t}$$

静态，介电体 → $\rho_f = 0$, $\vec{J}_f = 0$

$$\begin{cases} \nabla\cdot\vec{D} = 0 & ①' \\ \nabla\times\vec{E} = 0 & ②' \end{cases} \quad \begin{cases} \nabla\cdot\vec{B} = 0 & ① \\ \nabla\times\vec{H} = 0 & ② \end{cases}$$

+ Boundary condition

⇓

$$\begin{cases} \int\vec{D}\cdot d\vec{s} = 0 \\ \int\vec{E}\cdot d\vec{l} = 0 \end{cases} \quad \begin{cases} \int\vec{B}\cdot d\vec{s} = 0 \\ \int\vec{H}\cdot d\vec{l} = 0 \end{cases}$$

$\int\vec{B}\cdot d\vec{s} = 0$, $\vec{B}_\perp$ is continuous

$\int\vec{H}\cdot d\vec{l} = 0$, $\vec{H}_\parallel$ is continuous

由积分形式可给出边界条件

由②:

$$\vec{H} = -\nabla\varphi$$

由①: $\nabla\cdot(\mu_0\vec{H} + \mu_0\vec{M}) = 0$

$$\nabla\cdot\vec{H} + \nabla\cdot\vec{M} = 0$$

$$\Longrightarrow \nabla^2\varphi = \nabla\cdot\vec{M}$$

同样：

由②', $\vec{E} = -\nabla\varphi$

由①': $\nabla\cdot(\varepsilon_0\vec{E} + \vec{P}) = 0$

$$\Longrightarrow \nabla^2\varphi = \frac{1}{\varepsilon_0}\nabla\cdot\vec{P}$$

最一般且有效的方法：Green-function method

Good function: $u(\vec{r})$, $v(\vec{r})$

$$\int_\Sigma u\nabla v\cdot d\vec{s} = \int_T \nabla\cdot(u\nabla v)\,dV$$

$$= \int_T \nabla u\cdot\nabla v\,dV + \int_T u\nabla^2 v\,dV \quad ①$$

同样：

$$\int_\Sigma v\nabla u\,d\vec{s} = \int_T \nabla u\cdot\nabla v\,dV + \int_T v\nabla^2 u\,dV \quad ②$$

① - ② 可得：

$$\oint_\Sigma \left(u\frac{dv}{dn} - v\frac{du}{dn}\right) ds = \int_T \left(u\nabla^2 v - v\nabla^2 u\right) dV$$

Green ~~function~~ formula

若知道：$\nabla^2 u(\vec{r}) = f(\vec{r})$

$$\nabla^2 v(\vec{r}-\vec{r}_0) = \delta(\vec{r}-\vec{r}_0) \quad (\text{Further discussion})$$

$$v(\vec{r}-\vec{r}_0) = -\frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}_0|} \rightarrow (\text{下两页})$$

$$\Rightarrow \int_\Sigma \left(u(\vec{r})\frac{\partial v(\vec{r}-\vec{r}_0)}{\partial n} - v(\vec{r}-\vec{r}_0)\frac{\partial u(\vec{r})}{\partial n}\right) ds$$

$$= \int_T \left(u\,\delta(\vec{r}-\vec{r}_0) + \frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}_0|}f(\vec{r})\right) dV$$

考虑一个带电荷处在原点处.

$$\rho = Q\,\delta(\vec{r})$$

$$\Rightarrow \nabla\cdot\vec{E} = \frac{Q\,\delta(\vec{r})}{\epsilon_0}$$

$$\Rightarrow -\nabla^2\phi = \frac{Q\,\delta(\vec{r})}{\epsilon_0}$$

有公式: $\nabla^2\frac{1}{r} = -4\pi\,\delta(\vec{r})$

用 $r=r_0$ 的小球包围.

证明: $\int\nabla^2\frac{1}{r}\,dV = -4\pi\int\delta(\vec{r})\,dV = -4\pi$

$$= \int\nabla\cdot\left(\nabla\frac{1}{r}\right)dV = \int\nabla\frac{1}{r}\cdot d\vec{s}$$

$$\nabla\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}}\hat{x} + \cdots$$

$$= -\frac{1}{2}\frac{2x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{x} + \cdots$$

$$= -\frac{\vec{r}}{r^3} = -\frac{\vec{e_r}}{r^2} \rightsquigarrow \nabla\frac{1}{r} = -\frac{\vec{e_r}}{r^2}$$

$$= -\int\frac{\vec{e_r}\cdot d\vec{s}}{r^2} = -4\pi$$

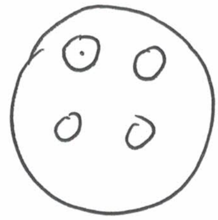
$$\Rightarrow \phi = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$$

$$\vec{E} = -\nabla\phi = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\vec{e_r}$$ ✓

很多电荷:

$$-\nabla^2\phi = \frac{\sum_i Q_i\,\delta(\vec{r}-\vec{r}_i)}{\epsilon_0}$$

$$\phi(\vec{r}) = \sum_i \frac{Q_i}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}_i|}$$



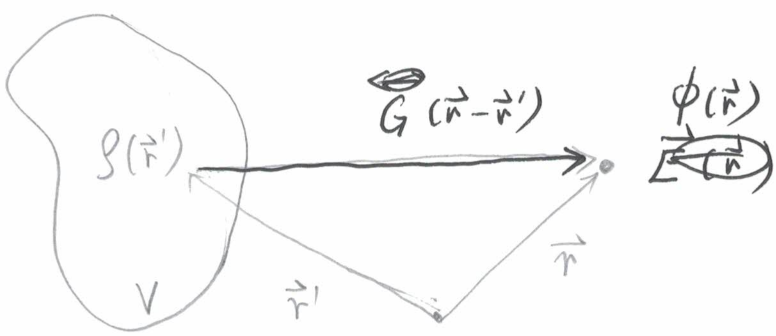
电荷连续分布 $Q_i = \rho(\vec{r}_i)\Delta V_i$

$$= \sum_i \frac{\rho(\vec{r}_i)\Delta V}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}_i|}$$

$$= \int dV'\frac{\rho(\vec{r}')}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}'|}$$

显然,给定电荷分布即求出电势,进一步可求出电场.

但实际情况是还要求出电荷分布,即环境对电场的响应.



Green function: $G(\vec{r}-\vec{r}')$

$$\phi(\vec{r}) = \int G(\vec{r}-\vec{r}')\,\rho(\vec{r}')\,dV'$$

$$\rightsquigarrow\ G(\vec{r}-\vec{r}') = \frac{1}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|}$$ 自由空间,无环境.

$$\Longrightarrow u(\vec{r}_0) = -\frac{1}{4\pi}\int \frac{f(\vec{r})}{|\vec{r}-\vec{r}_0|} - \int_\Sigma \left(u(\vec{r}) \frac{\partial v(\vec{r}-\vec{r}_0)}{\partial n} - v(\vec{r}-\vec{r}_0)\frac{\partial u(\vec{r})}{\partial n}\right) dS$$

让 Σ 是无穷远的一个积分，$u(|\vec{r}| \to \infty) \to 0$.

$$\Longrightarrow u(\vec{r}_0) = -\frac{1}{4\pi}\int \frac{f(\vec{r})}{|\vec{r}-\vec{r}_0|} dV + \frac{1}{4\pi}\int_\Sigma \frac{1}{|\vec{r}-\vec{r}_0|} \frac{\partial u(\vec{r})}{\partial n} ds$$

$$\Longrightarrow u(\vec{r}) = -\frac{1}{4\pi}\int \frac{f(\vec{r}\,')}{|\vec{r}\,'-\vec{r}|} dV' + \frac{1}{4\pi}\int_\Sigma \frac{1}{|\vec{r}-\vec{r}\,'|} \frac{\partial u(\vec{r}\,')}{\partial n'} ds'$$

It is strange that at both sides, the function of $u(\vec{r})$ appears. Only if we know $\frac{\partial u}{\partial n}$ at the boundary, we can solve the function exactly.

对于我们刚才考虑的静电极化、静磁问题

$f(\vec{r}) = \vec{M}(\vec{r})$,

$\nabla \psi = -\vec{H}(\vec{r})$

$$\Longrightarrow \psi(\vec{r}) = -\frac{1}{4\pi}\int_T \frac{\nabla' \cdot \vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} dV' - \underbrace{\frac{1}{4\pi}\int_\Sigma \frac{\vec{H}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\cdot d\vec{S}\,'}$$

作业：证明这一项为0。

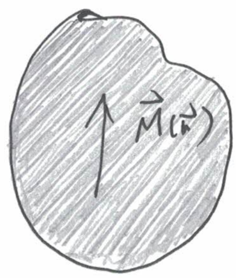
$$\varphi(\vec{r}) = -\frac{1}{4\pi}\int_{T\to\infty}\frac{\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}dv'$$

探究：

$$\varphi(\vec{r}) = -\frac{1}{4\pi}\int_{T\to\infty}\frac{\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}dv'$$

是否满足边界条件？

考虑处于一定空间范围内的 $\vec{M}$



$\nabla\cdot\vec{M}(\vec{r})$ 在边界处不连续

边界条件是什么？

$$\boxed{\vec{H}(\vec{r}) = -\nabla\varphi(\vec{r})}$$

①, $\vec{B}_\perp$ is continuous, $\vec{B} = \mu_0(\vec{H}+\vec{M})$

②, $\vec{H}_\parallel$ is continuous $\rightsquigarrow$ $\boxed{\varphi(\vec{r}_+) = \varphi(\vec{r}_-)}$ ①

从①可得：

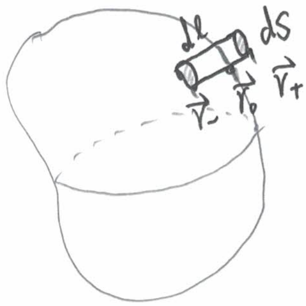
$$\vec{H}_\perp(\vec{r}_-) + \vec{M}_\perp(\vec{r}_-) = \vec{H}_\perp(\vec{r}_+)$$

$$\boxed{-\frac{\partial\varphi(\vec{r}_-)}{\partial n} + \vec{M}(\vec{r}_-)\cdot\vec{n} = -\frac{\partial\varphi(\vec{r}_+)}{\partial n}} \quad ②.$$

两个边界条件.

条件① 说明 φ 是个 good function

但是条件② 却难以证明.（怎么办）



$$\int \vec{H}\cdot d\vec{s} = \int \nabla\cdot\vec{H}\, dV = -\int \nabla^2\varphi\, ds\, dl$$

$$= -\int ds \int_{\vec{r}_-}^{\vec{r}_s} dl\, \nabla^2\varphi - \int ds \int_{\vec{r}_s}^{\vec{r}_+} dl\, \nabla^2\varphi$$

$$\boxed{\nabla^2\varphi = \nabla\cdot\vec{M}}$$

$$= -\int ds \int_{\vec{r}_-}^{\vec{r}_s} dl\, \nabla\cdot\vec{M} + 0$$

$$= -\int d\tilde{V}\, \nabla\cdot\vec{M} = -\int d\vec{s}\cdot\vec{M}$$

应用: Homogeneous magnetization.

Starting point: $\varphi(\vec{r}) = -\frac{1}{4\pi}\int_{T\to\infty} \frac{\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|} dV'$

The magnetization in the sphere:

$\vec{M}(r,\theta,\phi) = \vec{M}_0\,\Theta(a-r)$ → Heavy 阶越函数 $\Theta(x) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases}$

⇓

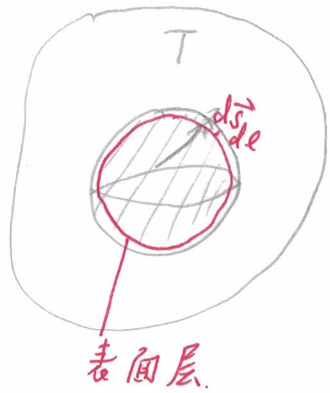
$r<a$, 均匀磁矩（为什么可以？）

$r>a$, 零磁矩.

$\vec{M}$ 产生的磁场：} demagnetization field

$\vec{P}$ 产生的电场：}

先进行一些数学运算，再结合数学图像让公式"说话"，形成物理图象，再根据物理图象对复杂现象进行判断.



$$\varphi(\vec{r}) = -\frac{1}{4\pi}\int_{T=\infty} \frac{\nabla'\cdot\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d\vec{r}\,'$$

$$= -\frac{1}{4\pi}\int_{V_-} \frac{\nabla'\cdot\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d\vec{r}\,'$$

$$-\frac{1}{4\pi}\int_{表面层} d\vec{s}\,' de' \frac{\nabla'\cdot\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

这一项着重考虑一下

$$= -\frac{1}{4\pi}\int_{表面} ds' \frac{\int de' \,\nabla'\cdot\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$\rightarrow -\vec{M}(\vec{r}\,')\cdot\vec{n}$

$$\Longrightarrow \varphi(\vec{r}) = -\frac{1}{4\pi}\int_{V_-} \frac{\nabla'\cdot\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d\vec{r}\,'$$

$$+\frac{1}{4\pi}\oint_{表面} ds' \frac{\vec{M}(\vec{r}\,')\cdot\vec{n}\,'}{|\vec{r}-\vec{r}\,'|}$$

$\Longrightarrow$ The generated magnetic field by $\vec{M}$: $\vec{H}_d(\vec{r}) = -\nabla\varphi$

补充:

$$\varphi(\vec{r}) = -\frac{1}{4\pi}\int_{T\to\infty} \frac{\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}'$$

$$= \frac{1}{4\pi}\int_{T\to\infty}\left(\nabla'\frac{1}{|\vec{r}-\vec{r}'|}\right)\cdot\vec{M}(\vec{r}')\,d\vec{r}'$$

$$= -\frac{1}{4\pi}\int_{T\to\infty}\left(\nabla\frac{1}{|\vec{r}-\vec{r}'|}\right)\cdot\vec{M}(\vec{r}')\,d\vec{r}'$$

$$= -\frac{1}{4\pi}\partial_\beta\int_{T\to\infty}\frac{1}{|\vec{r}-\vec{r}'|}M_\beta(\vec{r}')\,d\vec{r}'$$

若 M_β 均匀 →

$$= \left(-\frac{1}{4\pi}\partial_\beta\int_V\frac{1}{|\vec{r}-\vec{r}'|}\,d\vec{r}'\right)M_\beta$$

$$H^\alpha_{dem}(\vec{r}) = -\frac{\partial}{\partial x^\alpha}\varphi(\vec{r})$$

$$= \left(\frac{1}{4\pi}\partial_\alpha\partial_\beta\int_V\frac{1}{|\vec{r}-\vec{r}'|}\,d\vec{r}'\right)M_\beta$$

$$= -N_{\alpha\beta}M_\beta$$

⇝ 定义 $N_{\alpha\beta} = -\frac{1}{4\pi}\partial_\alpha\partial_\beta\int_V\frac{1}{|\vec{r}-\vec{r}'|}\,d\vec{r}'$

$N_{\alpha\beta}$ 称为 demagnetization factor. ⟹ 张量 $\begin{pmatrix} N_{xx} & N_{xy} & N_{xz} \\ N_{yx} & N_{yy} & N_{yz} \\ N_{zx} & N_{zy} & N_{zz} \end{pmatrix}$

性质: ①. $N_{\alpha\beta} = N_{\beta\alpha}$ ⇝ 实对称矩阵.

②. $Tr(N_{\alpha\beta}) = 1$, if $\vec{r}$ is in the volume.

$$Tr(N_{\alpha\beta}) = -\frac{1}{4\pi}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\int_V \frac{1}{|\vec{r}-\vec{r}'|} d\vec{r}'$$

$$= -\frac{1}{4\pi}\nabla^2 \int_V \frac{1}{|\vec{r}-\vec{r}'|} d\vec{r}'$$

$$= -\frac{1}{4\pi}\int_V \nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} d\vec{r}' = \begin{cases} 1, & \text{if } \vec{r} \text{ within the volume} \\ 0, & \text{if } \vec{r} \text{ is outside of the volume} \end{cases}$$

($\nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\,\delta(\vec{r}-\vec{r}')$)

③. $N_{\alpha\beta}$ 在很多场合非常有用，也可用于近似，因此在工程领域做成了表格

定义: $f(\vec{r}) = \frac{1}{4\pi}\int_V \frac{1}{|\vec{r}-\vec{r}'|} d\vec{r}'$ ⇝ 对一些好的几何形式，大家进行了计算.

$$N_{\alpha\beta} = \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} f(\vec{r})$$

例: 球体 (非常有用) .. 半径为 a

$\vec{r}$ 沿 z 轴

$$f(\vec{r}) = -\frac{1}{4\pi}\int_{sphere} \frac{1}{|\vec{r}-\vec{r}'|} d\vec{r}'$$

$$= -\frac{1}{4\pi}\int_0^a r'^2 dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}$$

($\sin\theta' d\theta' = -d\cos\theta'$; $\int_0^{2\pi} d\phi' = 2\pi$)

$$= -\frac{1}{2}\int_0^a r'^2 dr' \int_{-1}^1 dx \frac{1}{\sqrt{r^2+r'^2-2rr'x}}$$

$$= -\frac{1}{2}\int_0^a r'^2 dr' \quad \sqrt{r^2+r'^2-2rr'x} \; \frac{1}{-rr'} \Big|_{-1}^{1}$$

$$= \frac{1}{2r}\int_0^a dr' \, r' \left(\sqrt{r^2+r'^2-2rr'} - \sqrt{r^2+r'^2+2rr'} \right)$$

$$= \frac{1}{2r}\int_0^a dr' \, r' \left(\sqrt{(r-r')^2} - \sqrt{(r+r')^2} \right)$$

$$= \underbrace{\frac{1}{2r}\int_0^a dr' \, r' \, |r-r'|} - \frac{1}{2r}\int_0^a dr' \, r' \, (r+r')$$

分为 $r>r'$, $r<r'$ 两种情况

if $r<a$ ⟶

$$= \frac{1}{2r}\int_0^r dr' \, r'(r-r') + \frac{1}{2r}\int_r^a dr' \, r'(r'-r) - \frac{1}{2r}\int_0^a dr' \, r'(r+r')$$

$$= \frac{1}{2r}\int_0^r dr' \, (rr'-r'^2) + \frac{1}{2r}\int_r^a dr' \, (r'^2-rr') - \frac{1}{2r}\int_0^a dr' \, (rr'+r'^2)$$

$$= \frac{1}{2r}\left(r\frac{r'^2}{2} - \frac{r'^3}{3} \right)\Big|_0^r + \frac{1}{2r}\left(\frac{r'^3}{3} - \frac{1}{2}rr'^2 \right)\Big|_r^a$$

$$- \frac{1}{2r}\left(r\frac{r'^2}{2} + \frac{r'^3}{3} \right)\Big|_0^a$$

$$= \frac{1}{2r}\left(\frac{r^3}{2} - \frac{r^3}{3} \right) + \frac{1}{2r}\left(\frac{a^3}{3} - \frac{1}{2}ra^2 \right) - \frac{1}{2r}\left(\frac{r^3}{3} - \frac{r^3}{2} \right)$$

$$- \frac{1}{2r}\left(r\frac{a^2}{2} + \frac{a^3}{3} \right)$$

$$= \frac{r^2}{6} - \frac{a^2}{2} \qquad \text{if } r < a$$

if $r > a$

$$f(\vec{r}) = \frac{1}{2r}\int_0^a dr'\, r'\, |r - r'| - \frac{1}{2r}\int_0^a dr'\, r'(r + r')$$

$$= \frac{1}{2r}\int_0^a dr'\, r'(r - r') - \frac{1}{2r}\int_0^a dr'\, r'(r + r')$$

$$= -\frac{1}{2r}\int_0^a dr'\, 2r'^2 \qquad \text{~~constant~~}$$

$$= -\frac{1}{r}\frac{a^3}{3} = -\frac{1}{3}\frac{a^3}{r}$$

$$\Rightarrow f(\vec{r}) = \begin{cases} -\frac{1}{3}\frac{a^3}{r}, & r > a \\ \frac{r^2}{6} - \frac{a^2}{2}, & r < a \end{cases}$$

N_{xx}, N_{yy}, N_{zz} 容易计算.

① $\vec{r}$ is inside of the sphere:

$$N_{xx} = \frac{\partial}{\partial x^2}\left(\frac{1}{6}x^2 + \frac{1}{6}y^2 + \frac{z^2}{6} - \cdots\right) = \frac{1}{3}$$

$$N_{yy} = \frac{1}{3}$$

$$N_{zz} = \frac{1}{3}, \quad N_{\alpha \neq \beta} = 0.$$

② $\vec{r}$ is outside of the sphere.

$$N_{xx} = -\frac{a^3}{3} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2+y^2+z^2}} = -\frac{a^3}{3} \frac{3x^2-r^2}{r^5}$$

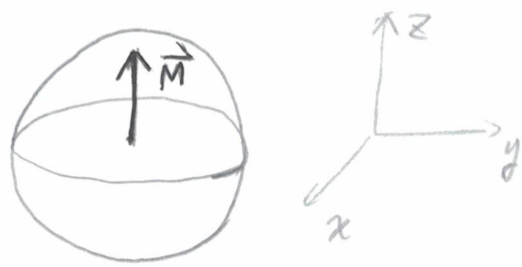
demagnetization factor:

$$N_{\alpha\beta} = -\frac{a^3}{r^5} \begin{pmatrix} 3x^2-r^2 & 3xy & 3xz \\ 3xy & 3y^2-r^2 & 3yz \\ 3xz & 3yz & 3z^2-r^2 \end{pmatrix}$$

磁场分布:

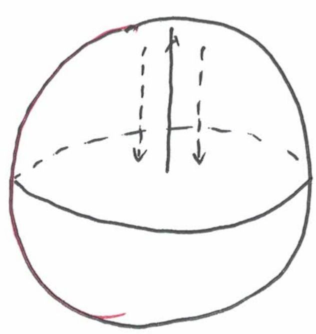
$$H_\alpha(\vec{r}) = -\boxed{N_{\alpha\beta}} M_\beta$$

$\vec{M} \rightarrow$ 沿z，不失一般性



$$\begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = -\begin{pmatrix} N_{xx} & N_{xy} & N_{xz} \\ N_{xz} & N_{yy} & N_{yz} \\ N_{xz} & N_{yz} & N_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ M_z \end{pmatrix}$$

$\vec{r}$ is inside of the sphere:



$H_x = 0$, $H_y = 0$, $H_z = -\frac{1}{3} M_z$

$\vec{r}$ is outside of the sphere:

$$H_x = \frac{a^3}{r^5} 3xz\, M_z \rightsquigarrow \frac{1}{r^3} \text{ decay.}$$

Question: ①. Only the surface term contributes.

$$\varphi(\vec r) = -\frac{1}{4\pi}\int_V \frac{\nabla'\cdot\vec M(\vec r')}{|\vec r-\vec r'|}dv' + \frac{1}{4\pi}\int_{Surface}\frac{\vec n\cdot\vec M(\vec r')}{|\vec r-\vec r'|}ds'$$

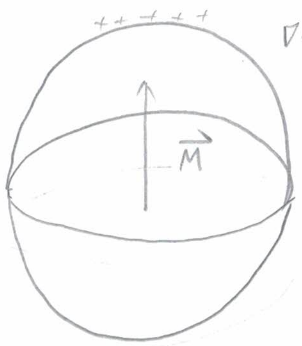
对比一下电荷情况

$$\rho(\vec r) = Q\,\delta(\vec r)$$

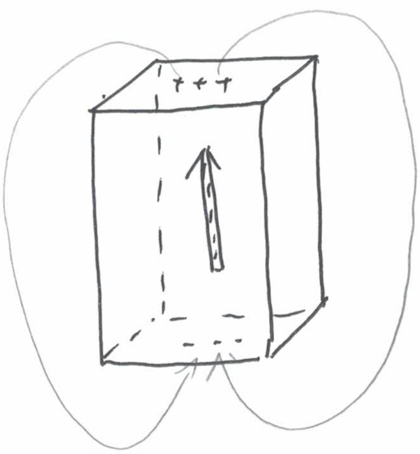
$$\begin{cases} \nabla^2\varphi = \dfrac{\rho(\vec r)}{\epsilon_0} = \dfrac{Q\,\delta(\vec r)}{\epsilon_0} \\ \nabla^2\varphi = \nabla\cdot\vec M \\ \nabla^2\varphi = \dfrac{\nabla\cdot\vec P}{\epsilon_0} \end{cases}$$

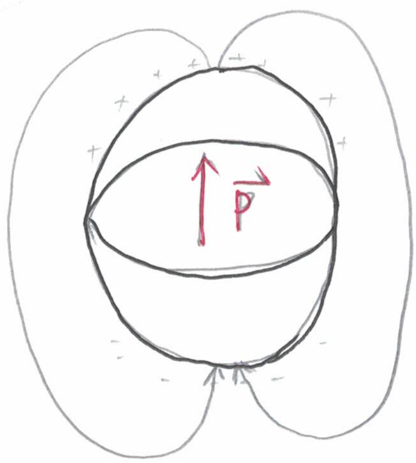
⇝ 右边都可翻成某种"荷"

$\nabla\cdot\vec M(\vec r)$ 分布 (δ 函数)

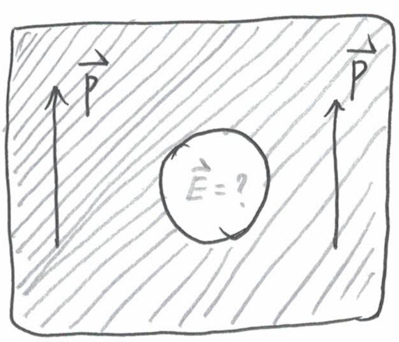


静磁学 → "磁荷"





作业:



手稿:

介质中 Maxwell 方程组

$$\nabla\cdot\vec{D}=\rho_f$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{H}=\vec{j}_f+\frac{\partial\vec{D}}{\partial t}$$

$$\vec{D}=\varepsilon_0\vec{E}+\vec{P}$$

$$\vec{H}=\frac{\vec{B}}{\mu_0}-\vec{M}$$

+ 边界条件.（由积分形式得到）

⇝ 静态问题（场不随时间变化）

$$\nabla\cdot\vec{D}=\rho_f$$

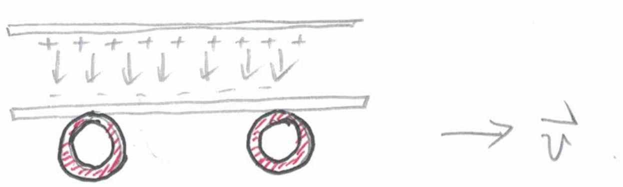
$$\nabla\times\vec{E}=0$$

⇝ 电、磁问题是分离的.

$$\nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{H}=\vec{j}_f$$

[静态问题 → 换个匀速运动的坐标系会怎么样?]



上一节，我们假设 ρ_f, $\vec{j}_f=0$, 但是有 $\vec{P}$，有 $\vec{M}$，求解了电场，磁场分布. 这一节，我们假设 $\vec{P}$, $\vec{M}=0$, 但允许 ρ_f, $\vec{j}_f$.

⇝ 那么我们研究的对象就成了 ρ_f, $\vec{j}_f$ 产生 $\vec{E}$, $\vec{B}$ 的问题.

$$\nabla\cdot\vec{E} = \rho_f/\varepsilon_0$$
$$\nabla\times\vec{E} = 0$$

⇝ $\vec{E} = -\nabla\varphi$, 泊松方程 $\nabla^2\varphi = -\frac{\rho_f}{\varepsilon_0}$

$$\nabla\cdot\vec{B} = 0$$
$$\nabla\times\vec{B} = \mu_0\vec{j}_f$$

⇝ $\vec{B} = \nabla\times\vec{A}$ ⇝ $\nabla\times(\nabla\times\vec{A}) = \mu_0\vec{j}_f$

$$= \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

总可以找到 $\vec{A}'$, 使 $\nabla\cdot\vec{A}' = 0$ ⟹ 规范条件

⇝ $\nabla^2\vec{A} = -\mu_0\vec{j}_f$ ⟹ 泊松方程.

$$\varphi(\vec{r}) = \frac{1}{4\pi}\int\frac{\rho_f(\vec{r}')/\varepsilon_0}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

$$\vec{A}(\vec{r}) = \frac{1}{4\pi}\int\frac{\mu_0\vec{j}_f(\vec{r}')}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

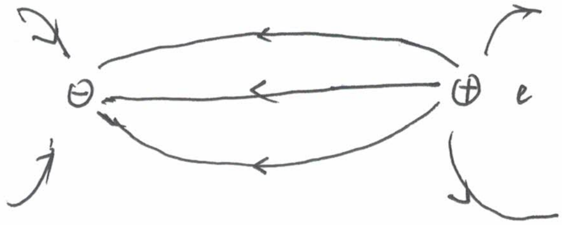
静态问题也得到解决!

However

材料是有 dynamics 的. $\vec{M}(\vec{r})$, $\vec{P}(\vec{r})$, ρ_f, $\vec{j}_f$ 本身也受到 $\vec{E}$, $\vec{B}$ 的影响.

第一步: 已知 ρ_f, $\vec{j}_f$ 求出电场, 磁场

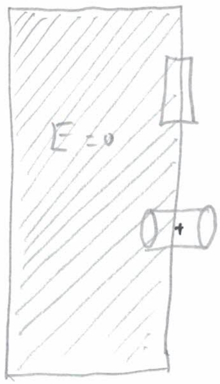
第二步: 自洽问题.



有

导体表面的电场

导体：静态时内部无电场



$\oint \vec{E}\cdot d\vec{l}=0 \Longrightarrow \vec{E}_{切}=0$

$\oint_A \vec{E}\cdot d\vec{a}=\frac{Q}{\varepsilon_0} \longrightarrow \vec{E}_{法}=\frac{\sigma_{面}}{\varepsilon_0}$

高斯定理

$\longrightarrow \vec{E}_{法}=\frac{\sigma_{面}}{2\varepsilon_0}$ （$\sigma_{面}$也很多情况下不知道的）

An interesting question:

我在本科时用电动力学+(一点扩散的知识)解决实验中的一个问题：

① 溶液中有正、负离子 ⊕ ⊖

带电

② 金属板放在溶液中会形成双电层

人

} 总需要知道一些学科的 facts.

$$\Longrightarrow u(\vec{r}_0) = -\frac{1}{4\pi}\int \frac{f(\vec{r})}{|\vec{r}-\vec{r}_0|} - \int_\Sigma \left(u(\vec{r})\frac{\partial V(\vec{r}-\vec{r}_0)}{\partial n} - V(\vec{r}-\vec{r}_0)\frac{\partial u(\vec{r})}{\partial n}\right) dS$$

让 Σ 是无穷远的一个积分，，$u(|\vec{r}| \to \infty) \to 0$.

$$\Longrightarrow u(\vec{r}_0) = -\frac{1}{4\pi}\int \frac{f(\vec{r})}{|\vec{r}-\vec{r}_0|} dV + \frac{1}{4\pi}\int_\Sigma \frac{1}{|\vec{r}-\vec{r}_0|} \frac{\partial u(\vec{r})}{\partial n} dS$$

$$\Longrightarrow u(\vec{r}) = -\frac{1}{4\pi}\int \frac{f(\vec{r}')}{|\vec{r}'-\vec{r}|} dV' + \frac{1}{4\pi}\int_\Sigma \frac{1}{|\vec{r}-\vec{r}'|} \frac{\partial u(\vec{r}')}{\partial n'} dS'$$

It is strange that at both sides, the function of $u(\vec{r})$ appears. Only if we know $\frac{\partial u}{\partial n}$ at the boundary, we can solve the function exactly.

对于我们刚才考虑的静电极化、静磁问题

$f(\vec{r}) = \vec{M}(\vec{r})$,

$\nabla\psi = -\vec{H}(\vec{r})$

$$\leadsto \psi(\vec{r}) = -\frac{1}{4\pi}\int_T \frac{\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|} dV' - \underbrace{\frac{1}{4\pi}\int_\Sigma \frac{\vec{H}(\vec{r}')}{|\vec{r}-\vec{r}'|}\cdot d\vec{S}'}$$

作业：证明这一项为0.

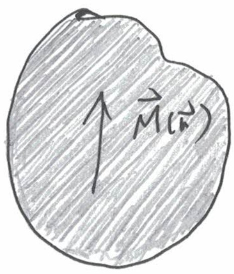
$$\varphi(\vec r) = -\frac{1}{4\pi}\int_{T\to\infty}\frac{\nabla'\cdot\vec M(\vec r')}{|\vec r-\vec r'|}\,dv'$$

探究:

$$\varphi(\vec r) = -\frac{1}{4\pi}\int_{T\to\infty}\frac{\nabla'\cdot\vec M(\vec r')}{|\vec r-\vec r'|}\,dv'$$

是否满足边界条件？

考虑处于一定空间范围内的 $\vec M$



$\nabla\cdot\vec M(\vec r)$ 在边界处不连续

边界条件是什么？

$$\boxed{\vec H(\vec r) = -\nabla\varphi(\vec r)}$$

①, $\vec B_\perp$ is continuous, $\vec B = \mu_0(\vec H + \vec M)$

②, $\vec H_\parallel$ is continuous $\rightsquigarrow$ $\boxed{\varphi(\vec r_+) = \varphi(\vec r_-)}$ ①

从① 可得:

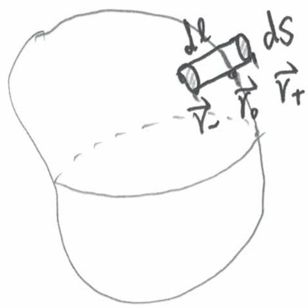
$$\vec H_\perp(\vec r_-) + \vec M_\perp(\vec r_-) = \vec H_\perp(\vec r_+)$$

$$\boxed{-\frac{\partial\varphi(\vec r_-)}{\partial n} + \vec M(\vec r_-)\cdot\vec n = -\frac{\partial\varphi(\vec r_+)}{\partial n}}$$ ②.

两个边界条件.

条件① 说明 φ 是个 good function

但是条件② 却难以证明.（怎么办）



$$\int \vec{H}\cdot d\vec{s} = \int \nabla\cdot\vec{H}\, dV = -\int \nabla^2\varphi\, ds\, dl$$

$$= -\int ds \int_{\vec{r}_-}^{\vec{r}_s} dl\, \nabla^2\varphi - \int ds \int_{\vec{r}_s}^{\vec{r}_+} dl\, \nabla^2\varphi$$

$$\boxed{\nabla^2\varphi = \nabla\cdot\vec{M}}$$

$$= -\int ds \int_{\vec{r}_-}^{\vec{r}_s} dl\, \nabla\cdot\vec{M} + 0$$

$$= -\int d\tilde{V}\, \nabla\cdot\vec{M} = -\int d\vec{s}\cdot\vec{M}$$

应用: Homogeneous magnetization.

Starting point: $\varphi(\vec{r}) = -\frac{1}{4\pi}\int_{T\to\infty} \frac{\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\, dv'$

The magnetization in the sphere:

$$\vec{M}(r,\theta,\phi) = \vec{M}_0\,\Theta(a-r)$$ → Heavy 阶越函数 $\Theta(x)=\begin{cases}1, & x>0\\ 0, & x<0\end{cases}$

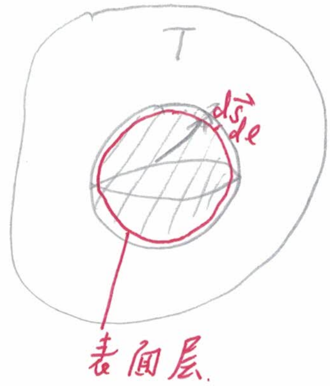
⇓

$r<a$, 指向磁矩（为什么可以？）

$r>a$, 零磁矩.

$\vec{M}$ 产生的磁场：} demagnetization field
$\vec{P}$ 产生的电场：}

先进行一些数学运算，再结合数学图象让公式"说话"，形成物理图象，再根据物理图象对更复杂现象进行判断.



$$\varphi(\vec{r}) = -\frac{1}{4\pi}\int_{T=\infty} \frac{\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}'$$

$$= -\frac{1}{4\pi}\int_{V_-} \frac{\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}'$$

$$-\frac{1}{4\pi}\int_{\text{表面层}} d\vec{s}'\,dl' \frac{\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

↓ 这一项着重考虑一下

$$= -\frac{1}{4\pi}\int_{\text{表面}} ds' \frac{\int dl'\,\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|} \quad \rightarrow -\vec{M}(\vec{r}')\cdot\vec{n}'$$

$$\Longrightarrow \varphi(\vec{r}) = -\frac{1}{4\pi}\int_{V_-} \frac{\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}' + \frac{1}{4\pi}\oint_{\text{表面}} ds' \frac{\vec{M}(\vec{r}')\cdot\vec{n}'}{|\vec{r}-\vec{r}'|}$$

$\Longrightarrow$ The generated magnetic field by $\vec{M}$: $\vec{H}_d(\vec{r}) = -\nabla\varphi$

计算:

$$\phi(\vec{r}) = -\frac{1}{4\pi}\int_{T\to\infty} \frac{\nabla'\cdot\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d\vec{r}\,'$$

$$= \frac{1}{4\pi}\int_{T\to\infty} \left(\nabla' \frac{1}{|\vec{r}-\vec{r}\,'|}\right)\cdot\vec{M}(\vec{r}\,')\, d\vec{r}\,'$$

$$= -\frac{1}{4\pi}\int_{T\to\infty} \left(\nabla \frac{1}{|\vec{r}-\vec{r}\,'|}\right)\cdot\vec{M}(\vec{r}\,')\, d\vec{r}\,'$$

$$= -\frac{1}{4\pi}\,\partial_\beta \int_{T\to\infty} \frac{1}{|\vec{r}-\vec{r}\,'|} M_\beta(\vec{r}\,')\, d\vec{r}\,'$$

若 M_β 均匀

$$= \left(-\frac{1}{4\pi}\,\partial_\beta \int_V \frac{1}{|\vec{r}-\vec{r}\,'|}\, d\vec{r}\,'\right) M_\beta$$

$$H^\alpha_{dem}(\vec{r}) = -\frac{\partial}{\partial x^\alpha}\phi(\vec{r})$$

$$= \left(\frac{1}{4\pi}\,\partial_\alpha\partial_\beta \int_V \frac{1}{|\vec{r}-\vec{r}\,'|}\, d\vec{r}\,'\right) M_\beta$$

$$= -N_{\alpha\beta} M_\beta$$

⇝ 定义了 $N_{\alpha\beta} = -\frac{1}{4\pi}\,\partial_\alpha\partial_\beta \int_V \frac{1}{|\vec{r}-\vec{r}\,'|}\, d\vec{r}\,'$

$N_{\alpha\beta}$ 称为 demagnetization factor. ⟹ 张量 $\begin{pmatrix} N_{xx} & N_{xy} & N_{xz} \\ N_{yx} & N_{yy} & N_{yz} \\ N_{zx} & N_{zy} & N_{zz} \end{pmatrix}$

性质: ①. $N_{\alpha\beta} = N_{\beta\alpha}$ ⇝ 实对称矩阵.

② $\mathrm{Tr}(N_{\alpha\beta}) = 1$, if $\vec{r}$ is in the volume.

$$\mathrm{Tr}(N_{\alpha\beta}) = -\frac{1}{4\pi}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\int_V \frac{1}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

$$= -\frac{1}{4\pi}\nabla^2\int_V \frac{1}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

$$= -\frac{1}{4\pi}\int_V \nabla^2\frac{1}{|\vec{r}-\vec{r}'|}d\vec{r}' \quad \left(\nabla^2\frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\,\delta(\vec{r}-\vec{r}')\right)$$

$$= \begin{cases} 1, & \text{if } \vec{r} \text{ within the volume} \\ 0, & \text{if } \vec{r} \text{ is outside of the volume.} \end{cases}$$

③. $N_{\alpha\beta}$ 在很多场合非常有用，也可用于近似，因此在工程领域做成了表格

定义：$f(\vec{r}) = \frac{1}{4\pi}\int_V \frac{1}{|\vec{r}-\vec{r}'|}d\vec{r}'$ ⇝ 对一些好的几何形式，大家进行了计算。

$$N_{\alpha\beta} = \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}f(\vec{r})$$

例：球体（非常有用）．．半径为a

$\vec{r}$ 沿z轴

$$f(\vec{r}) = -\frac{1}{4\pi}\int_{\text{sphere}}\frac{1}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

$$= -\frac{1}{4\pi}\int_0^a r'^2 dr'\int_0^\pi \sin\theta' d\theta'\int_0^{2\pi}d\phi'\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}$$

($\sin\theta' d\theta' = -d\cos\theta'$; $\int_0^{2\pi}d\phi' = 2\pi$)

$$= -\frac{1}{2}\int_0^a r'^2 dr'\int_{-1}^1 dx\,\frac{1}{\sqrt{r^2+r'^2-2rr'x}}$$

$$= -\frac{1}{2}\int_0^a r'^2 dr' \sqrt{r^2+r'^2-2rr'x}\ \frac{1}{-rr'}\Big|_{-1}^{1}$$

$$= \frac{1}{2r}\int_0^a dr'\, r'\left(\sqrt{r^2+r'^2-2rr'} - \sqrt{r^2+r'^2+2rr'}\right)$$

$$= \frac{1}{2r}\int_0^a dr'\, r'\left(\sqrt{(r-r')^2} - \sqrt{(r+r')^2}\right)$$

$$= \underbrace{\frac{1}{2r}\int_0^a dr'\, r'|r-r'|}_{} - \frac{1}{2r}\int_0^a dr'\, r'(r+r')$$

分为 $r>r'$ 两种情况
$r<r'$

if $r<a$ →

$$= \frac{1}{2r}\int_0^r dr'\, r'(r-r') + \frac{1}{2r}\int_r^a dr'\, r'(r'-r) - \frac{1}{2r}\int_0^a dr'\, r'(r+r')$$

$$= \frac{1}{2r}\int_0^r dr'(rr'-r'^2) + \frac{1}{2r}\int_r^a dr'(r'^2-rr') - \frac{1}{2r}\int_0^a dr'(rr'+r'^2)$$

$$= \frac{1}{2r}\left(r\frac{r'^2}{2}-\frac{r'^3}{3}\right)\Big|_0^r + \frac{1}{2r}\left(\frac{r'^3}{3}-\frac{1}{2}rr'^2\right)\Big|_r^a$$

$$-\frac{1}{2r}\left(r\frac{r'^2}{2}+\frac{r'^3}{3}\right)\Big|_0^a$$

$$= \frac{1}{2r}\left(\frac{r^3}{2}-\frac{r^3}{3}\right) + \frac{1}{2r}\left(\frac{a^3}{3}-\frac{1}{2}ra^2\right) - \frac{1}{2r}\left(\frac{r^3}{3}-\frac{r^3}{2}\right)$$

$$-\frac{1}{2r}\left(r\frac{a^2}{2}+\frac{a^3}{3}\right)$$

$$= \frac{r^2}{6} - \frac{a^2}{2} \qquad \text{if } r < a$$

if $r > a$

$$f(\vec{r}) = \frac{1}{2r}\int_0^a dr' r' |r - r'| - \frac{1}{2r}\int_0^a dr' r'(r + r')$$

$$= \frac{1}{2r}\int_0^a dr' r'(r - r') - \frac{1}{2r}\int_0^a dr' r'(r + r')$$

$$= -\frac{1}{2r}\int_0^a dr' 2r'^2 =$$

~~constant~~

$$= -\frac{1}{r}\frac{a^3}{3} = -\frac{1}{3}\frac{a^3}{r}$$

$$\rightsquigarrow f(\vec{r}) = \begin{cases} -\frac{1}{3}\frac{a^3}{r}, & r > a \\ \frac{r^2}{6} - \frac{a^2}{2}, & r < a \end{cases}$$

N_{xx}, N_{yy}, N_{zz} 容易计算.

① . $\vec{r}$ is inside of the sphere:

$$N_{xx} = \frac{\partial}{\partial x^2}\left(\frac{1}{6}x^2 + \frac{1}{6}y^2 + \frac{z^2}{6} - \cdots\right) = \frac{1}{3}$$

$$N_{yy} = \frac{1}{3}$$

$$N_{zz} = \frac{1}{3}, \quad N_{\alpha \neq \beta} = 0.$$

② $\vec{r}$ is outside of the sphere.

$$N_{xx} = -\frac{a^3}{3}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}} = -\frac{a^3}{3}\frac{3x^2-r^2}{r^5}$$

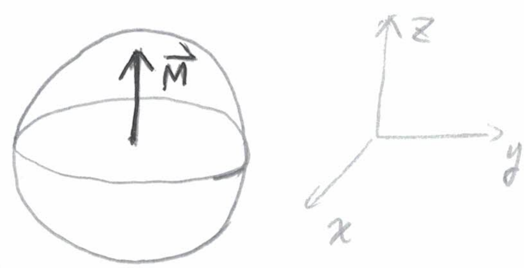
demagnetization factor:

$$n_{\alpha\beta} = -\frac{a^3}{r^5}\begin{pmatrix} 3x^2-r^2 & 3xy & 3xz \\ 3xy & 3y^2-r^2 & 3yz \\ 3xz & 3yz & 3z^2-r^2 \end{pmatrix}$$

磁场分布:

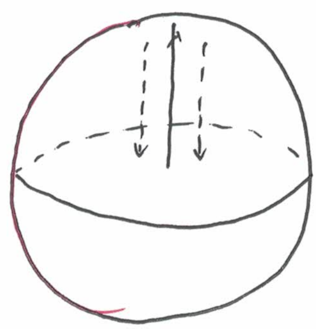
$$H_\alpha(\vec{r}) = -\boxed{N_{\alpha\beta}}\, M_\beta$$

$\vec{M} \rightarrow$ 沿z, 不失一般性



$$\begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = -\begin{pmatrix} N_{xx} & N_{xy} & N_{xz} \\ N_{xz} & N_{yy} & N_{yz} \\ N_{xz} & N_{yz} & N_{zz} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ M_z \end{pmatrix}$$

$\vec{r}$ is inside of the sphere:



$H_x = 0$, $H_y = 0$, $H_z = -\frac{1}{3} M_z$

$\vec{r}$ is outside of the sphere:

$H_x = \frac{a^3}{r^5} 3xz\, M_z \rightsquigarrow \frac{1}{r^3}$ decay.

Question: Q. ①. Only the surface term contributes.

$$\varphi(\vec{r}) = -\frac{1}{4\pi}\int_{V} \frac{\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}dv' + \frac{1}{4\pi}\int_{Surface} \frac{\vec{n}\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}ds'$$

对比一下电荷情况

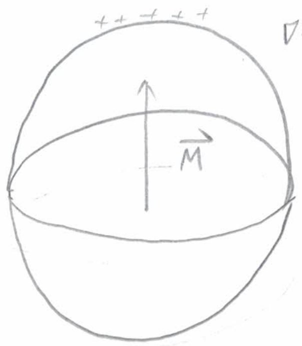
$$\rho(\vec{r}) = Q\,\delta(\vec{r})$$

$$\nabla^2\phi = \frac{\rho(\vec{r})}{\epsilon_0} = \frac{Q\,\delta(\vec{r})}{\epsilon_0}$$

$$\nabla^2\varphi = \nabla\cdot\vec{M}$$

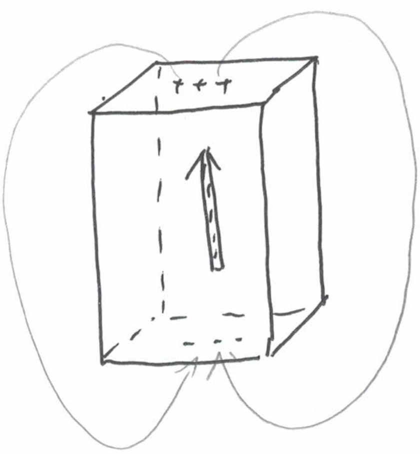
$$\nabla^2\varphi = \frac{\nabla\cdot\vec{P}}{\epsilon_0}$$

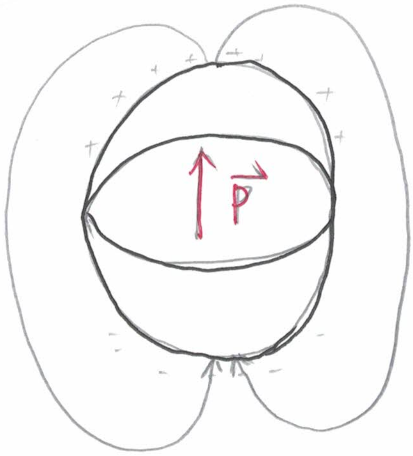
⟿ 右边都可翻成某种"荷"



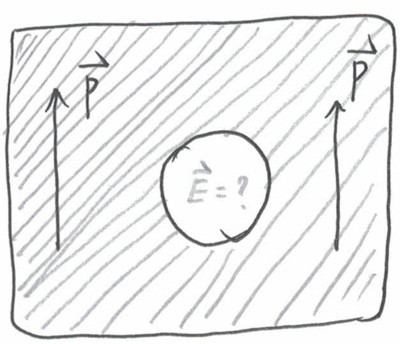
$\nabla\cdot\vec{M}(\vec{r})$ 分布 (δ 函数)

静磁学 → "磁荷"





作业:



小结:

介质中 Maxwell 方程组

$$\nabla\cdot\vec{D} = \rho_f$$

$$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{H} = \vec{j}_f + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

+ 边界条件 (由积分形式得到)

⟿ 静态问题 (场不随时间变化)

$$\nabla\cdot\vec{D} = \rho_f$$

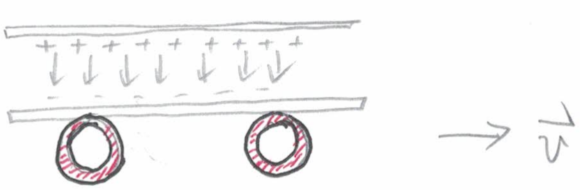
$$\nabla\times\vec{E} = 0$$

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{H} = \vec{j}_f$$

⟿ 电、磁问题是分离的.

[静态问题 → 换个匀速运动的坐标系会怎么样?]



上一节,我们将 ρ_f, $\vec{j}_f = 0$, 但是有 $\vec{P}$, 有 $\vec{M}$, 求解了电场, 磁场分布. 这一节, 我们将 $\vec{P}$, $\vec{M} = 0$, 只允许 ρ_f, $\vec{j}_f$.

⇝ 那么我们研究的对象就成了 ρ_f, $\vec{j}_f$ 与 $\vec{E}$, $\vec{B}$ 的问题.

$$\nabla\cdot\vec{E}=\rho_f/\varepsilon_0$$

$$\nabla\times\vec{E}=0$$

⇝ $\vec{E}=-\nabla\varphi$, 泊松方程 $\nabla^2\varphi=-\frac{\rho_f}{\varepsilon_0}$

$$\nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{B}=\mu_0\vec{j}_f$$

⇝ $\vec{B}=\nabla\times\vec{A}$ ⇝ $\nabla\times(\nabla\times\vec{A})=\mu_0\vec{j}_f$

$$\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$$

总可以找到 $\vec{A}'$, 使 $\nabla\cdot\vec{A}'=0$ ⟹ 规范条件

⇝ $\nabla^2\vec{A}=-\mu_0\vec{j}_f$ ⟹ 泊松方程.

$$\varphi(\vec{r})=\frac{1}{4\pi}\int\frac{\rho_f(\vec{r}')/\varepsilon_0}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

$$\vec{A}(\vec{r})=\frac{1}{4\pi}\int\frac{\mu_0\vec{j}_f(\vec{r}')}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

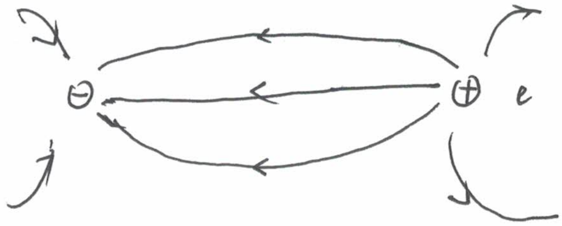
} 静态问题也得到解决!

However

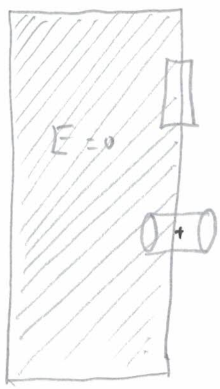
材料是有 dynamics 的. $\vec{M}(\vec{r})$, $\vec{P}(\vec{r})$, ρ_f, $\vec{j}_f$ 本身也受到 $\vec{E}$, $\vec{B}$ 的影响.

第一步: 已知 ρ_f, $\vec{j}_f$ 求出电场, 磁场

第二步: 自洽问题.



导体表面的电场. 导体：静态时内部无电场



$\oint \vec{E}\cdot d\vec{l} = 0 \Rightarrow \vec{E}_{切} = 0$

$\oint_A \vec{E}\cdot d\vec{a} = \frac{Q}{\varepsilon_0} \longrightarrow \vec{E}_{法} = \frac{\sigma_{面}}{\varepsilon_0}$

高斯定理

$\rightarrow \vec{E}_{法} = \frac{\sigma_{面}}{2\varepsilon_0}$ （$\sigma_{面}$也很多情况下不知道的）

An interesting question:

我在本科时用电动力学+（一点扩散的知识）解决实验中的一个问题：

① 溶液中有正、负离子 ⊕ ⊖

带电 ② 金属板放在溶液中会形成双电层

只需要知道一些学科的facts.

$\phi(\vec{r})$

离子的空间分布服从玻尔兹曼分布

$$n_\pm(\vec{r}) = n_0 e^{\mp e\phi(\vec{r})/(k_BT)}$$

$$\approx n_0 \mp n_0 e\phi(\vec{r})/(k_BT)$$

↓

n_0为溶液中离子浓度.

由泊松方程

$$\nabla^2\phi(\vec{r}) = -\frac{\rho}{\varepsilon_0}$$

$$\rho = n_+ - n_- = n_0 e^{-e\phi(\vec{r})/(k_BT)} - n_0 e^{e\phi(\vec{r})/(k_BT)}$$

$$\Longrightarrow \nabla^2\phi(\vec{r}) = \frac{n_0}{\varepsilon_0}\left(e^{\frac{e\phi(\vec{r})}{k_BT}} - e^{-\frac{e\phi(\vec{r})}{k_BT}}\right)$$

$$\approx \frac{2n_0 e\phi(\vec{r})}{\varepsilon_0 k_BT}$$

(在一些特殊情况可以线性化)

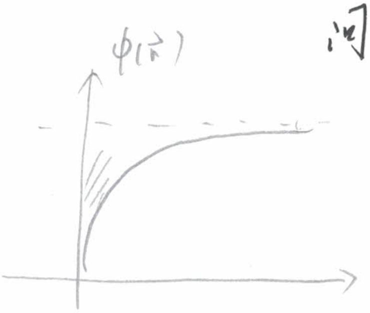
令 $k^2 = \dfrac{2n_0 e}{k_BT\varepsilon_0}$

⟹ 线性化后的方程

$$\nabla^2\phi(\vec{r}) = k^2\phi(\vec{r})$$

求出 $\phi(\vec{r})$ ⟹ 溶液里面放入少量(带电)的纳米颗粒,

问:这些纳米颗粒能否因电作用被吸附?



边界条件: $E_{法} = \dfrac{\sigma_e}{\varepsilon_0}$

长直导线问题

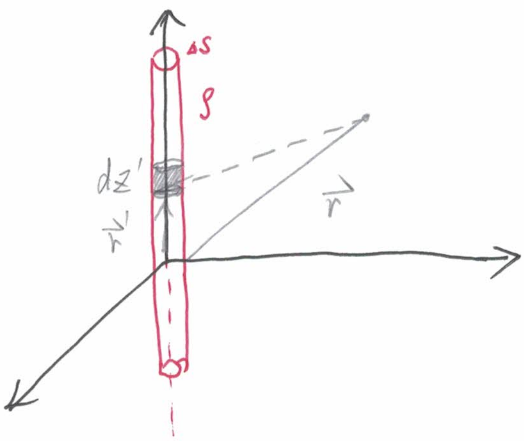
①细长直导线（无限长，比如说）电荷线密度为 ρ，求电势、电场

How?

②细长直导线（无限长，比如说）电荷流密度为 $\vec{j}$，求磁矢势、磁场

How?

有限长的问题如何求解？ ⇝ 作业



$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int dz' ds' \frac{\rho_f(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

推广积分：

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int dz' dx' dy' \frac{\rho_f(\vec{r}')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\int dz' dx' dy' \frac{\rho_f(z')\,\delta(x')\,\delta(y')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\int dz' \frac{\rho_f(z')}{\sqrt{x^2+y^2+(z-z')^2}} = \frac{\rho_f}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty} dz' \frac{1}{\sqrt{x^2+y^2+(z-z')^2}}$$

$$= \frac{\rho_f}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} dz' \frac{1}{\sqrt{r^2 + z'^2}}$$

$r^2 = x^2 + y^2$

⟶ 初步可以判断电势 φ 并不依赖于 z，仅依赖于 (x, y)

$$= \frac{\rho_f}{4\pi\varepsilon_0} \ln\left(z' + \sqrt{r^2 + z'^2}\right)\Big|_{-\frac{L}{2}}^{\frac{L}{2}}, \quad L \to \infty$$

$$\frac{1 + \frac{1}{2}\frac{2z'}{(r^2+z'^2)^{\frac{1}{2}}}}{z' + \sqrt{r^2 + z'^2}} = \frac{1 + \frac{z'}{(r^2+z'^2)^{\frac{1}{2}}}}{z' + \sqrt{r^2+z'^2}} = \frac{1}{\sqrt{r^2+z'^2}}$$

$$\frac{\rho_f}{4\pi\varepsilon_0} \ln\left(\frac{\frac{L}{2} + \sqrt{r^2 + (L/2)^2}}{-\frac{L}{2} + \sqrt{r^2 + (L/2)^2}}\right)$$

至少 $L \gg r$

$$= \frac{\rho_f}{4\pi\varepsilon_0} \ln \frac{\frac{L}{2} + \frac{L}{2}\sqrt{1 + \left(\frac{2r}{L}\right)^2}}{-\frac{L}{2} + \frac{L}{2}\sqrt{1 + \left(\frac{2r}{L}\right)^2}}$$

$$= \frac{\rho_f}{4\pi\varepsilon_0} \ln \frac{\frac{L}{2} + \frac{L}{2}\left(1 + \frac{1}{2}\left(\frac{2r}{L}\right)^2\right)}{-\frac{L}{2} + \frac{L}{2}\left(1 + \frac{1}{2}\left(\frac{2r}{L}\right)^2\right)}$$

$$= \frac{\rho_f}{4\pi\varepsilon_0} \ln \frac{L + r^2/L}{r^2/L} = \frac{\rho_f}{4\pi\varepsilon_0} \ln\left(\frac{L^2}{r^2}\right)$$

$$= \frac{\rho_f}{2\pi\varepsilon_0} \ln \frac{L}{r}$$

$L \to \infty$ φ ~~[illegible]~~ 都 $\to \infty$

∇：柱坐标 $\nabla f = \frac{\partial f}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial f}{\partial \varphi}\vec{e_\varphi} + \frac{\partial f}{\partial z}\vec{e_z}$

$$\vec{E} = -\nabla\phi$$

$$= -\frac{\rho_f}{2\pi\varepsilon_0}\frac{r}{L}\left(-\frac{L}{r^2}\right)\vec{e_r} = +\boxed{\frac{\rho_f}{2\pi\varepsilon_0}\frac{1}{r}\vec{e_r}}$$

同样对于 载流导线.

$$\vec{A}(\vec{r}) = \frac{1}{4\pi}\int \frac{\mu_0 \vec{j_f}(\vec{r}')}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

$\vec{j_f}$ 沿 z： $\vec{j_f}(\vec{r}') = j_f\vec{z}'\,\delta(x')\,\delta(y')$

$$A_z(\vec{r}) = \frac{\mu_0 j_f}{4\pi}\int \frac{\delta(x')\,\delta(y')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}dx'dy'dz'$$

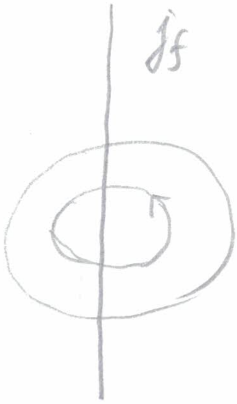
$$= \frac{\mu_0 j_f}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} dz' \frac{1}{\sqrt{r^2+(z-z')^2}}$$

$$= \frac{\mu_0 j_f}{2\pi}\ln\frac{L}{r}$$

$$\vec{B} = \nabla\times\vec{A} = \frac{1}{r}\begin{vmatrix} \vec{e_r} & r\vec{e_\varphi} & \vec{e_z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z(r) \end{vmatrix} = \frac{1}{r}\left(r\vec{e_\varphi}\frac{\partial}{\partial r}A_z(r)\right)$$

$$= \vec{e_\varphi}\frac{\partial}{\partial r}A_z(r)$$

$$\Rightarrow \vec{B} = \frac{\mu_0 j_f}{2\pi}\vec{e}_\varphi \left(\frac{r}{L}\right)\left(-\frac{L}{r^2}\right) = -\frac{\mu_0 j_f}{2\pi r}\vec{e}_\varphi$$



"辐射"

Near-field

这些场有什么更深刻含义.(研究成果).

(全面描述).

Weyl identity 或者 Coulomb 积分.

$$I = \int d\vec{r}' \frac{e^{i\vec{k}\cdot\vec{\rho}'} f(x')}{|\vec{r}-\vec{r}'|} = \frac{2\pi}{|\vec{k}|} e^{i\vec{k}\cdot\vec{\rho}} \int dx' e^{-|x-x'||\vec{k}|} f(x')$$

$$\frac{e^{ik\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = \frac{i}{2\pi}\int dk_y dk_z \frac{e^{ik_x|x-x'|+ik_y(y-y')+ik_z(z-z')}}{k_x}$$

其中 $k = \sqrt{k_x^2+k_y^2+k_z^2}$ (证明:作业)

静态问题 $k=0$.

$$\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = \cdots ; \quad \sqrt{k_x^2+k_y^2+k_z^2} = 0$$

$$A_z(\vec{r},\omega) = \frac{\mu_0}{4\pi}\int d\vec{r}' \frac{J_z(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{\mu_0}{4\pi}\int dx'dy'\int_{-\infty}^{+\infty} dz' J_z(x',y') \frac{i}{2\pi}\int dk_y dk_z \frac{e^{ik_x|x-x'|+ik_y(y-y')+ik_z(z-z')}}{k_x}$$

$k_x \rightsquigarrow i\sqrt{k_y^2+k_z^2}$

$$\int_{-\infty}^{+\infty} dz' \, e^{-ik_z z'} = 2\pi\delta(k_z)$$

$$= \frac{\mu_0}{4\pi}\int dx'dy' J_z(x',y') \, i \int dk_y \frac{e^{-|k_y||x-x'|+ik_y(y-y')}}{i|k_y|}$$

但简单情况 $J_z\,\delta(x')\delta(y')$

$$= \frac{\mu_0}{4\pi} J_z \int dk_y \frac{e^{-|k_y||x|+ik_y y}}{|k_y|}$$

$x > 0$

$$= \frac{\mu_0}{4\pi} J_z \int dk_y \frac{e^{-|k_y|x+ik_y y}}{|k_y|}$$

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$$

$$\vec{B} = \nabla\times\vec{A} = \frac{\partial}{\partial y}A_z\,\hat{x} - \frac{\partial}{\partial x}A_z\,\hat{y} = B_x\,\hat{x} + B_y\,\hat{y}$$

$$B_x(\vec{r}) = \frac{\partial A_z}{\partial y} = \frac{\mu_0}{4\pi} J_z \int dk_y \frac{ik_y}{|k_y|}\left(e^{-|k_y|x+ik_y y}\right)$$

$$= \sum_{k_y} B_x(k_y)\, e^{ik_y y}$$

$$\rightsquigarrow B_x(k_y) = \frac{\mu_0}{2} J_z \frac{ik_y}{|k_y|}\left(e^{-|k_y|x}\right)$$

$$B_y(\vec{r}) = -\frac{\partial A_z}{\partial x} = -\frac{\mu_0}{4\pi} J_z \int dk_y \, e^{-|k_y|x + ik_y y}$$

$$\rightsquigarrow B_y(k_y) = -\frac{\mu_0}{2} J_z \, e^{-|k_y|x}$$

$$\begin{cases} B_x(k_y, x) = \dfrac{\mu_0}{2} J_z \dfrac{ik_y}{|k_y|} e^{-|k_y|x} \\ B_y(k_y, x) = -\dfrac{\mu_0}{2} J_z \, e^{-|k_y|x} \end{cases}$$

$$ik_y B_x(k_y, x) = |k_y| B_y(k_y, x)$$

$k_y > 0$ 时，$B_x(k_y, x) = -i B_y(k_y, x)$

$k_y < 0$ 时，$B_x(k_y, x) = i B_y(k_y, x)$

} 相位与动量锁定.

以上我们分析了源(电荷、电流；$\vec{M}$, $\vec{P}$)产生的电场、磁场问题，给出了一些一般性的求解方法。

接下来我们讨论一个也是直截了当的方法：求解偏微分方程。(建议大家用不同方法求解同一问题)

在无电荷分布时，电势 φ 满足 Laplace 方程

$$\nabla^2 \varphi = 0$$

其通解已经在数学物理方法中给出

我们将利用这些数学工具探究有趣的物理现象

导体的静电学、静磁学

上节已经讲过

$\nabla\cdot\vec{E} = \frac{\rho_f}{\varepsilon_0}$

$\nabla\times\vec{E} = 0$

$\Downarrow$

$\vec{E} = -\nabla\varphi$

$\nabla^2\varphi = -\frac{\rho_f}{\varepsilon_0}$

$\nabla\cdot\vec{B} = 0$

$\nabla\times\vec{B} = \mu_0\vec{j}_f$

$\Downarrow$

规范

$\vec{B} = \nabla\times\vec{A}$, $\nabla\cdot\vec{A} = 0$

$\nabla^2\vec{A} = -\mu_0\vec{j}_f$

若已知 ρ_f, $\vec{j}_f$，则 φ, $\vec{A}$，或 $\vec{E}$, $\vec{B}$ 是已知的.

$\varphi(\vec{r}) = \frac{1}{4\pi}\int\frac{\rho(\vec{r}')/\varepsilon_0}{|\vec{r}-\vec{r}'|}d\vec{r}'$

$\vec{A}(\vec{r}) = \frac{1}{4\pi}\int\frac{\mu_0\vec{j}_f(\vec{r}')}{|\vec{r}-\vec{r}'|}d\vec{r}'$

上面在无限长直导线中已经给出了例子.

⇒ 但是，物理世界的丰富之处在于，材料（导体、绝缘体）中 ρ_f, $\vec{j}_f$, $\vec{M}$, $\vec{P}$ 等本身有动力学，它们与电磁场的相互作用构成了我们随处可见的很多丰富现象。《电动力学》拟定量描述这些现象.

最简单的例子之一 即导体.

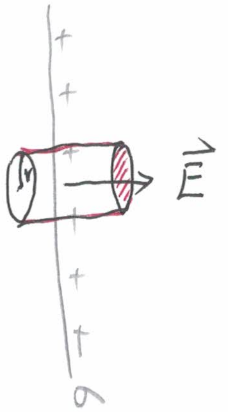
导体：有自由电子，在电场作用下这些电子就会运动.

1). 导体内部电场为0.

2). 导体内部不带净电荷，净电荷只能分布在导体表面上

$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} = 0.$

面密度为 σ



$$\vec{E}\cdot\pi r^2 = \frac{\sigma\cdot\pi r^2}{\epsilon_0}$$

$$\Longrightarrow E = \frac{\sigma}{\epsilon_0}$$

边界条件： $\boxed{\epsilon_0 \frac{\partial\varphi}{\partial n} = -\sigma}$

3). 导体表面上电场必沿法线方向。

因此导体是个等势体。

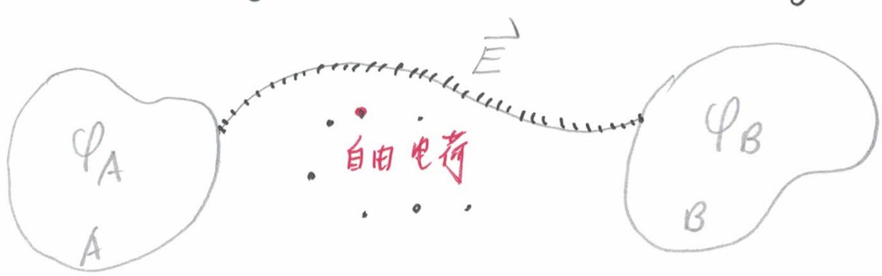
如何知道导体表面电荷密度 ⟶ 导体与环境相互作用

求解的方程：

导体内部，外部，Laplace 方程 $\nabla^2\varphi = 0$

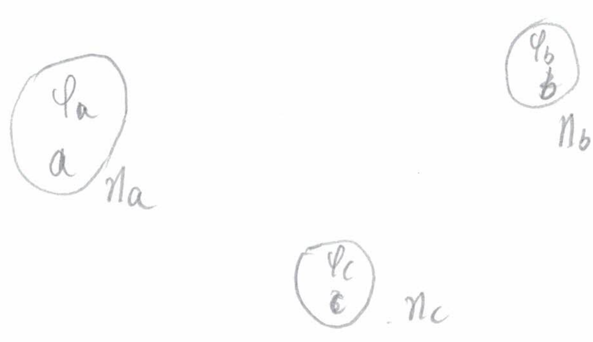
导体表面， $\epsilon_0 \frac{\partial\varphi}{\partial n} = -\sigma$

$$\int \vec{E}\cdot d\vec{l} = \int_A^B (-\nabla\varphi)\cdot d\vec{l} = \varphi_A - \varphi_B \neq 0.$$



$\varphi_A \neq \varphi_B$ 会发生什么

体系中静电场的总能量



在 a, b, c 上放置了 n_a, n_b, n_c 的净电荷

我们来求一下静电场的总能量

$$W=\frac{1}{2}\int_{T\to\infty}\vec{E}\cdot\vec{D}\,d\vec{r}$$

$$=-\frac{1}{2}\int_{T\to\infty}\nabla\varphi\cdot\vec{D}\,d\vec{r}=-\frac{1}{2}\int_{T\to\infty}\left[\nabla\cdot(\varphi\vec{D})-\varphi\nabla\cdot\vec{D}\right]d\vec{r}$$

$$=\frac{1}{2}\int\varphi\,\nabla\cdot\vec{D}\,d\vec{r}=\frac{1}{2}\int\varphi\,\rho_f\,d\vec{r}\overset{\text{导体}}{=}\frac{1}{2}\varphi Q_f$$

问：多个导体总能量.

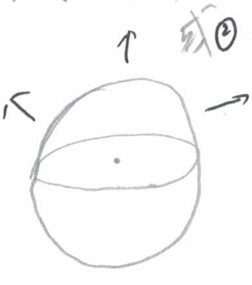
$$\frac{1}{2}\int\varphi(\vec{r})\rho_f(\vec{r})\,d\vec{r}=\frac{1}{2}\sum_{a=1}^{N}\varphi_a Q_a$$

问：Q_a 与 φ_a 是否有关系？

举个例子

一个导体，①φ 知道了，是否知道 Q？

或② Q 知道了，是否知道 φ



先考虑：

②：$\int\rho(\vec{r})\,d\vec{r}=Q$

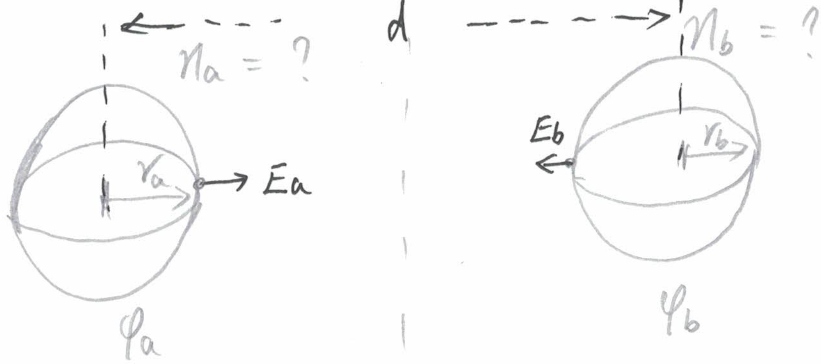
$$4\pi\int r^2dr\,\sigma\,\delta(r_a)=Q\implies\sigma=\frac{Q}{4\pi r_a^2}$$

$$E=\frac{\sigma}{\epsilon}=\frac{Q}{4\pi r_a^2\epsilon}\quad(\text{表面})$$

$$\int_{r_a}^{+\infty}-\nabla\varphi\,dr=\varphi(r_a)=\frac{Q}{4\pi\epsilon_0 r_a}$$

☆ $\frac{Q}{\epsilon_0}=E\cdot4\pi r^2\to E=\frac{Q}{4\pi\epsilon_0 r^2}=-\nabla\varphi$

两个球:



$$E_a = \frac{q_a}{4\pi r_a^2 \epsilon}, \qquad E_b = \frac{q_b}{4\pi r_b^2 \epsilon}$$

$\Longrightarrow$ 表现地像点电荷

$$\varphi_a = \frac{q_a}{4\pi\epsilon r_a}, \qquad \varphi_b = \frac{q_b}{4\pi\epsilon r_b}$$

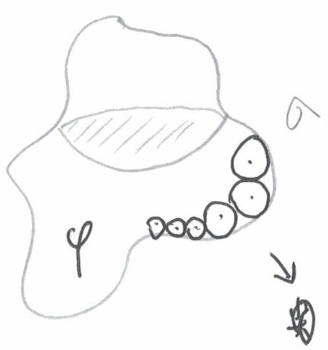
我们利用这个性质就能"定性"了解很多问题.

(1) 当 $\varphi_a = \varphi_b$ 时，即想象球"a"和球"b"以某种方式连接在一起，如导线，如点接触解

$$\frac{q_a}{4\pi\epsilon r_a} = \frac{q_b}{4\pi\epsilon r_b} \Longrightarrow \frac{q_a}{q_b} = \frac{r_a}{r_b}$$

$$\Longrightarrow \frac{\sigma_a}{\sigma_b} = \frac{r_a}{r_b}$$，电荷分布与"曲率"有关

怎么来严格证明这点？

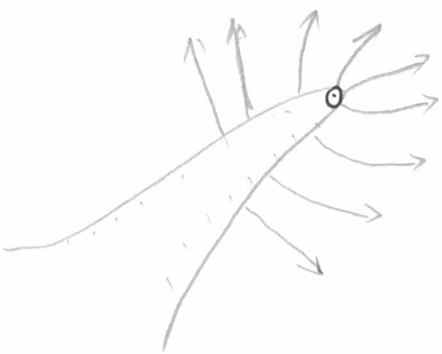


$\sigma(\vec{r})$，问已知 φ，$\sigma(\vec{r})$ 如何分布

(提问!)

(2). $\frac{E_a}{E_b} = \frac{n_a}{n_b}\frac{r_b^2}{r_a^2} = \frac{r_b}{r_a}$

$r_a \ll r_b \Longrightarrow E_a \gg E_b$ 尖端放电



导体是等势体.（$\nabla\cdot\vec{E}$，$\nabla\times\vec{E}$ 决定）

1) 刚才看到确定 φ 与 $n(\vec{r})$ 的关系，就确定了导体对外产生的电场.

2) 孤立球体非常简单 ⇝ 类似于"点电荷"

3). 体系的静电能量只与 φ_a，n_a 有关.

多个孤立导体问题

一个：$n_a = C\varphi_a$ （球：$n_a = 4\pi\epsilon r_a \varphi_a$）

多个：$n_a = \sum_{b=1}^{N} C_{ab}\varphi_b$，（重复 b 表示和）

C：电容，C_{ab} 互容

$$E_e = \frac{1}{2}\sum_{a=1}^{N}\varphi_a n_a = \frac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N}\varphi_a C_{ab}\varphi_b$$

$\vec{D} = \epsilon \vec{E} + \vec{P}$

$\Downarrow$ 导体的介电常数，还是环境的介电常数？

$\vec{D} = \epsilon_0 \vec{E}$

$\to$ 导体中无电场

$U_e = \frac{\epsilon_0}{2} \int_{T\to\infty} \vec{E} \cdot \vec{E} \, d\vec{r}$

$\delta \vec{E}$

$\Rightarrow \delta U_e = \epsilon_0 \int_{T\to\infty} \vec{E} \cdot \delta \vec{E} \, d\vec{r}$

空间电势的一个分布

$= \epsilon_0 \int_{T\to\infty} (-\nabla \varphi(\vec{r})) \cdot \delta \vec{E} \, d\vec{r}$

$= \epsilon_0 \int_{T\to\infty} \left[-\nabla \cdot (\varphi(\vec{r}) \delta \vec{E}) + \varphi(\vec{r}) \nabla \cdot \delta \vec{E} \right] d\vec{r}$

$\delta \rho_f(\vec{r}) / \epsilon_0$

$= \epsilon_0 \int_{T\to\infty} \varphi(\vec{r}) \, \nabla \cdot \delta \vec{E} \, d\vec{r}$

$= \int_{T\to\infty} \varphi(\vec{r}) \, \delta \rho_f(\vec{r}) \, d\vec{r}$

电荷密度

$= \sum_a \varphi_a \delta \eta_a$

同样: $\delta U_e = \sum_a \delta \varphi_a \eta_a$

$\hookrightarrow$ change in energy in terms of the change in the potentials of the conductor

$$= \frac{1}{2} (\varphi_1, \varphi_2, \cdots, \varphi_N) \begin{pmatrix} C_{11} & C_{12} & \cdots \\ C_{21} & C_{22} & \\ \vdots & & \ddots & C_{NN} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix}$$

C_{ab} 性质

① 若 a 上增加一无穷小 charge，δn_a

$$\varphi_b = \sum_{a=1}^{N} n_a C_{ab}^{-1}$$

$$\delta\varphi_b = \sum_{a=1}^{N} \delta n_a C_{ab}^{-1}$$

$$\Longrightarrow \boxed{C_{ab}^{-1} = \frac{\delta\varphi_b}{\delta n_a}}$$

② 若 a 上增加一无穷小 potential，$\delta\varphi_a$

$$n_a = \sum_{b=1}^{N} C_{ab}\varphi_b$$

$$\delta n_a = \sum_{b=1}^{N} C_{ab}\delta\varphi_b$$

$$\Longrightarrow \boxed{C_{ab} = \frac{\delta n_a}{\delta\varphi_b}}$$

即物理意义

继续考虑 ②.

$$E_e = \frac{1}{2}\sum_{a,b} (\varphi_a + \delta\varphi_a) C_{ab} (\varphi_b + \delta\varphi_b)$$

$$= \underbrace{\frac{1}{2}\sum_{a,b} \varphi_a C_{ab} \varphi_b}_{E_e^{(0)}} + \underbrace{\frac{1}{2}\sum_{a,b} \delta\varphi_a C_{ab}\varphi_b + \frac{1}{2}\sum_{a,b}\varphi_a C_{ab}\delta\varphi_b}_{\delta E_e}$$

$$U_e = \frac{1}{2}\int_{r\to\infty} \vec{E}\cdot\vec{D}\, d\vec{r}, \quad \delta E_e = \frac{1}{2}\int \delta\vec{E}\cdot\vec{D}\, d\vec{r} + \frac{1}{2}\int \vec{E}\cdot\delta\vec{D}\, d\vec{r}$$

$$\delta\varphi_a \longrightarrow \delta\vec{E},\ \delta\vec{D}$$

$$\to \varphi_a = \frac{\delta U_e}{\delta n_a}$$

$$n_a = \frac{\delta U_e}{\delta \varphi_a} = \sum_b C_{ab} \varphi_b$$

$$\Rightarrow C_{ab} = \frac{\delta n_a}{\delta \varphi_b} = \frac{\delta^2 U_e}{\delta \varphi_a \delta \varphi_b} = C_{ba}$$ $\Rightarrow$ 实对称矩阵（本征值 $\tilde{C}_{aa}$）

$$\Rightarrow U_e = \frac{1}{2}\sum_a \varphi_a n_a = \frac{1}{2}\sum_{a,b} \varphi_a C_{ab} \varphi_b \simeq \frac{1}{2}\sum \tilde{C}_{aa} \tilde{\varphi}_a^2$$

$$= \frac{1}{2}\sum_{a,b} n_a C_{ab}^{-1} n_b$$

（$\sum_{a,b} \varphi_a C_{ab} \varphi_b$ = 二次型）

性质：$C_{aa} > 0$，$C_{ab} < 0$（不讲）

Thomson's theorem:

The energy of the actual electrostatic field is minimum relative to the energies of fields which could be produced by any other distributions of the charges on or in the conductors.

证明：假定各个导体上的电荷总数保持不变，我们来改变一下分布 $\delta\rho_f(\vec{r})$

$$\delta U_e = \int_{T\to\infty} \varphi(\vec{r})\, \delta\rho_f(\vec{r})\, d\vec{r} = \sum_a \varphi_a \int_V \delta\rho_f^{(a)}(\vec{r})\, d\vec{r} = 0$$

$\leadsto$ 能量处于极值！

探索例:

$r \gg \{r_a, r_b\}$

$$\phi_1 = \frac{e_1}{C_1}, \quad \phi_2 = \frac{e_1}{4\pi\varepsilon_0 r} \quad (\text{点电荷})$$

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} \begin{pmatrix} \frac{e_1}{C_1} \\ \frac{e_1}{4\pi\varepsilon_0 r} \end{pmatrix}$$

$$\Longrightarrow \begin{cases} e_1 = C_{11} \dfrac{e_1}{C_1} + C_{12} \dfrac{e_1}{4\pi\varepsilon_0 r} \\ 0 = C_{12} \dfrac{e_1}{C_1} + C_{22} \dfrac{e_1}{4\pi\varepsilon_0 r} \end{cases}$$

$$\begin{pmatrix} 0 \\ e_1 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} \begin{pmatrix} \frac{e_1}{4\pi\varepsilon_0 r} \\ \frac{C_2}{e_1} \end{pmatrix}$$

$$\begin{cases} 0 = C_{11} \dfrac{e_1}{4\pi\varepsilon_0 r} + C_{12} \dfrac{e_1}{C_2} \\ e_1 = C_{12} \dfrac{e_1}{4\pi\varepsilon_0 r} + C_{22} \dfrac{e_1}{C_2} \end{cases}$$

$$\Longrightarrow \begin{cases} \dfrac{C_{11}}{C_1} + \dfrac{C_{12}}{4\pi\varepsilon_0 r} = 1 \quad ① \\ \dfrac{C_{12}}{C_1} + \dfrac{C_{22}}{4\pi\varepsilon_0 r} = 0 \quad ②, \quad \dfrac{C_{11}}{4\pi\varepsilon_0 r} + \dfrac{C_{12}}{C_2} = 0 \quad ③ \\ \dfrac{C_{12}}{4\pi\varepsilon_0 r} + \dfrac{C_{22}}{C_2} = 1 \quad ④ \end{cases}$$

由③，

$$C_{11} = -4\pi\varepsilon_0 r \frac{C_{12}}{C_2} \overset{\text{由①}}{=} C_1 - \frac{C_{12}}{4\pi\varepsilon_0 r} C_1$$

由②，

$$C_{22} = -4\pi\varepsilon_0 r \frac{C_{12}}{C_1} \overset{\text{由④}}{=} C_2 - \frac{C_{12}}{4\pi\varepsilon_0 r} C_2$$

$$\Longrightarrow C_{12} = \frac{4\pi\varepsilon_0 r\, C_1}{-(4\pi\varepsilon_0 r)^2/C_2 + C_1} = \frac{C_1 C_2}{-4\pi\varepsilon_0 r + \frac{C_1 C_2}{4\pi\varepsilon_0 r}}$$

$$\simeq -\frac{C_1 C_2}{4\pi\varepsilon_0 r - \frac{C_1 C_2}{4\pi\varepsilon_0 r}}$$

$$C_{11} = -4\pi\varepsilon_0 r \frac{C_1}{-4\pi\varepsilon_0 r + \frac{C_1 C_2}{4\pi\varepsilon_0 r}}$$

$$= \frac{C_1}{1 - \frac{C_1 C_2}{(4\pi\varepsilon_0 r)^2}} \approx C_1\left(1 + \frac{C_1 C_2}{(4\pi\varepsilon_0 r)^2}\right)$$

$$C_{22} \approx C_2\left(1 + \frac{C_1 C_2}{(4\pi\varepsilon_0 r)^2}\right) \Longrightarrow$$

$C_2 \gg C_1$，即 $r_b \gg r_a$

$C_1 \approx 4\pi\varepsilon_0 r_a$，$C_2 \approx 4\pi\varepsilon_0 r_b$

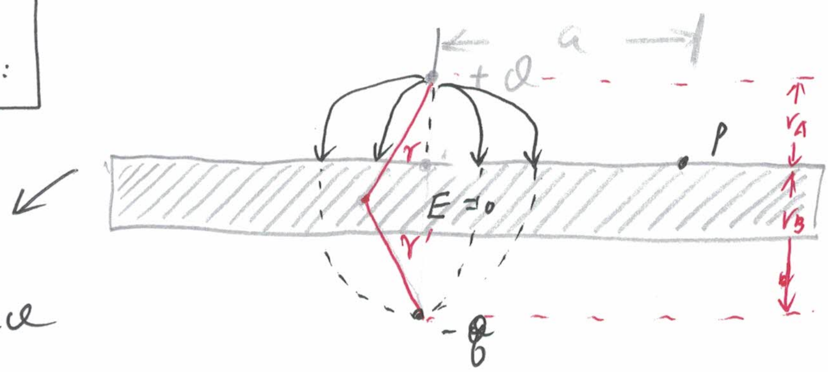
继续深入两个导体互作用问题之前，我们还可以考虑一个更极端的情况，来找些"感觉"

• Q

⇓

假定一导体非常小，我们可以用点电荷来表示

镜像法:



occupy half space

$d \gg \lambda$

(单层原子是否成立)

二维电子气?

$$\varphi = \frac{Q}{4\pi\varepsilon r} - \frac{q}{4\pi\varepsilon r'}$$

$$\frac{Q}{4\pi\varepsilon r_A} - \frac{q}{4\pi\varepsilon r_A'} = \frac{Q}{4\pi\varepsilon r} - \frac{q}{4\pi\varepsilon r'}$$

$$\Rightarrow \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_A} - \frac{1}{r}\right) = \frac{q}{4\pi\varepsilon}\left(\frac{1}{r_A'} - \frac{1}{r'}\right)$$

The principle of this method is to find fictitious point charges, which together with the given charge or charges, produces a field such that the surface of the conductor is an equipotential surface.

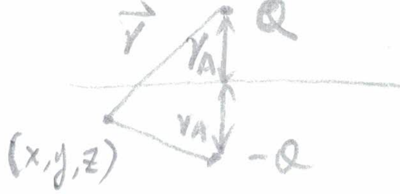
$$\varphi_p = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r_A^2 + a^2}} - \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r_B^2 + a^2}}$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_A} - \frac{q}{4\pi\varepsilon_0}\frac{1}{r_B}$$,对所有a成立

求 q, r_B

$a \gg r_A, r_B$, $\varphi_p \to 0$, $\frac{Q}{r_A} = \frac{q}{r_B}$, 选 $q = Q$, 即 $\varphi_p = 0$ 始终满足

$$\varphi_P \Big|_{\text{表面}} = 0$$



$$\vec{r}_A = r_A \vec{z}$$

$$\varphi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2+y^2+(z-r_A)^2}} - \frac{Q}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2+y^2+(z+r_A)^2}}$$

$$\sigma(\vec{r})\Big|_{z=0} = \varepsilon_0 \frac{\partial \varphi(\vec{r})}{\partial z}\Big|_{z=0}$$

$$= \varepsilon_0 \frac{Q}{4\pi\varepsilon_0} \left[\frac{1}{2} \frac{2(z-r_A)}{(x^2+y^2+(z-r_A)^2)^{\frac{3}{2}}} - \frac{1}{2} \frac{2(z+r_A)}{(x^2+y^2+(z-r_A)^2)^{\frac{3}{2}}} \right]_{z=0}$$

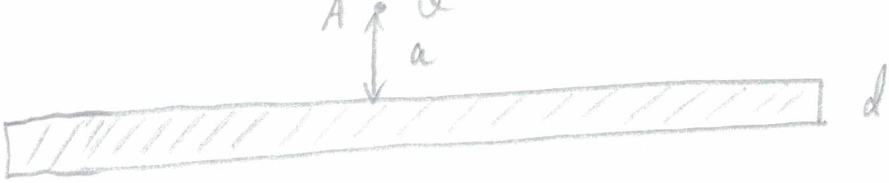
$$= \frac{Q}{4\pi} \frac{-2r_A}{(x^2+y^2+r_A^2)^{\frac{3}{2}}} = -\frac{Q}{2\pi} \frac{r_A}{(x^2+y^2+r_A^2)^{\frac{3}{2}}}$$

$$n = \int dxdy \ \sigma(\vec{r}) = -\frac{Q}{2\pi} \int_0^\infty r dr d\theta \frac{r_A}{(r^2+r_A^2)^{\frac{3}{2}}}$$

$$= -Q \int_0^\infty \frac{1}{2} dx \frac{r_A}{(x+r_A^2)^{\frac{3}{2}}}$$

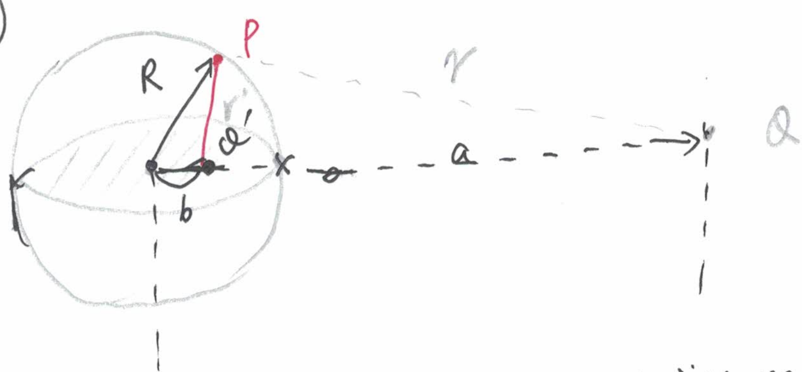
$$= -Q \left(-\frac{r_A}{(x+r_A^2)^{\frac{1}{2}}} \right) \Bigg|_0^\infty = -Q$$

作业:



求表面电荷分布 (不一定容易做!)

(孤立球)



"接地"

$$\varphi_P = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{Q'}{r'}\right) = 0$$

$$\frac{Q}{r} + \frac{Q'}{r'} = 0$$

$$\rightsquigarrow \begin{cases} \dfrac{Q}{a-R} + \dfrac{Q'}{R-b} = 0 \\[2ex] \dfrac{Q}{a+R} + \dfrac{Q'}{R+b} = 0 \end{cases} \qquad \rightsquigarrow \quad Q' = -\frac{R-b}{a-R}\,Q$$

$$\Downarrow$$

$$\frac{Q}{a+R} - \frac{1}{R+b}\,\frac{R-b}{a-R}\,Q = 0$$

$$\Longrightarrow \frac{a-R}{a+R} = \frac{R-b}{R+b} \Longrightarrow b = \frac{R^2}{a}$$

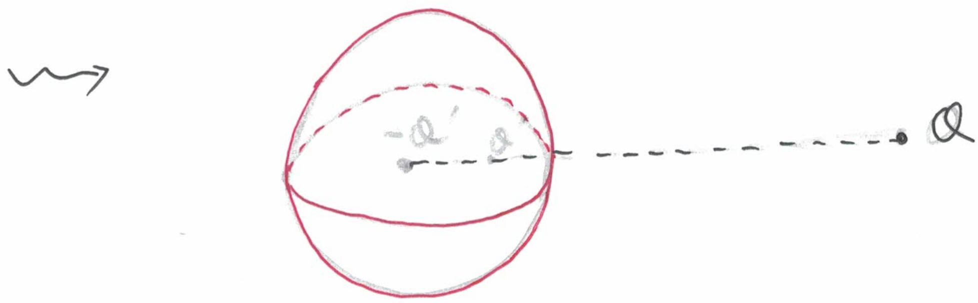
$$Q' = -\frac{R - \frac{R^2}{a}}{a-R}\,Q = -\frac{R}{a}\,\frac{a-R}{a-R}\,Q = -\frac{R}{a}\,Q$$

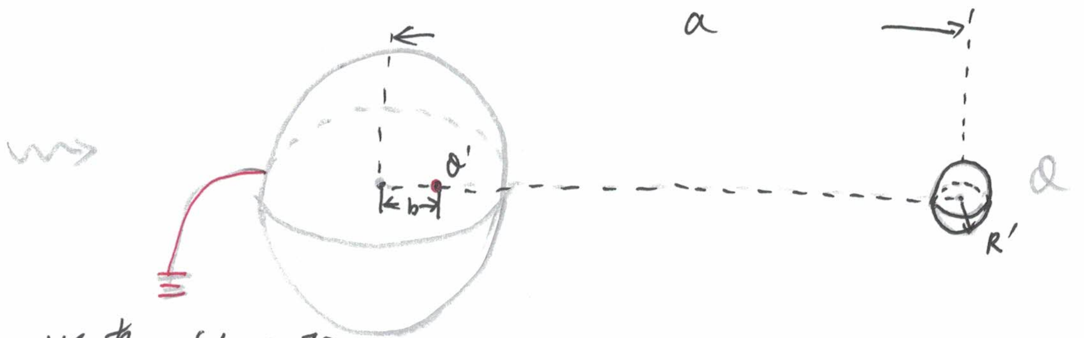
(检查: $\dfrac{a-R}{a+R} \Big| \quad \dfrac{R-\frac{R^2}{a}}{R+\frac{R^2}{a}} = \dfrac{1-\frac{R}{a}}{1+\frac{R}{a}} = \dfrac{a-R}{a+R}$ ✓)

再在球中心放在有效电荷$-Q'$，则总的电荷变又保持为0。

同时球的电势 $\varphi = -\dfrac{4\pi Q'}{R} = \dfrac{4\pi}{R}\dfrac{R}{a}Q = \dfrac{4\pi}{a}Q$

保持等电势

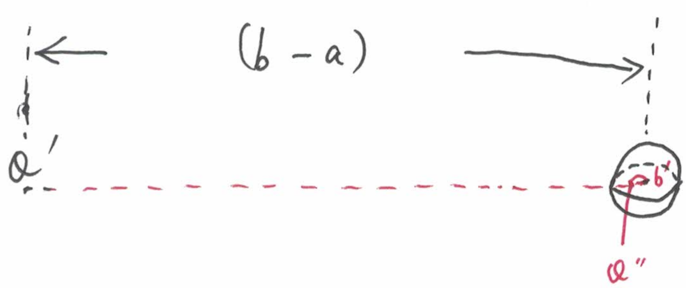




探索：(级数解)

① 先假定Q均匀分布 → $Q' = -\dfrac{R}{a}Q$，$b = \dfrac{R^2}{a}$

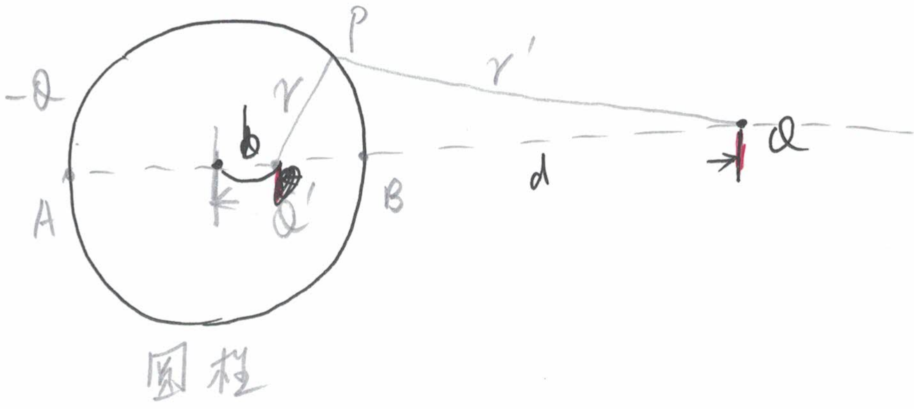
② Q' 作为点电荷 影响 Q 的分布，再用镜像法考虑



$b' = \dfrac{R'^2}{b-a}$，$\tilde{Q}' = -\dfrac{R'}{b-a}Q' = \dfrac{R'}{b-a}\dfrac{R}{a}Q$

$\tilde{Q}'' = Q - \tilde{Q}' = Q - \dfrac{R'}{b-a}\dfrac{R}{a}Q$ 在球心

还可以用镜像法？



$\varphi = 2Q \ln \frac{L}{r}$，不是 $\frac{1}{r}$ 那样的

$\varphi_P = 2Q' \ln \frac{L}{r} + 2Q \ln \frac{L}{r'}$

$Q' = -Q$ （证明）. ⟿ 可以用来构造严格解

$= 2Q \ln \frac{r}{r'}$

$\varphi_A = 2Q \ln \frac{R+b}{R+d}$

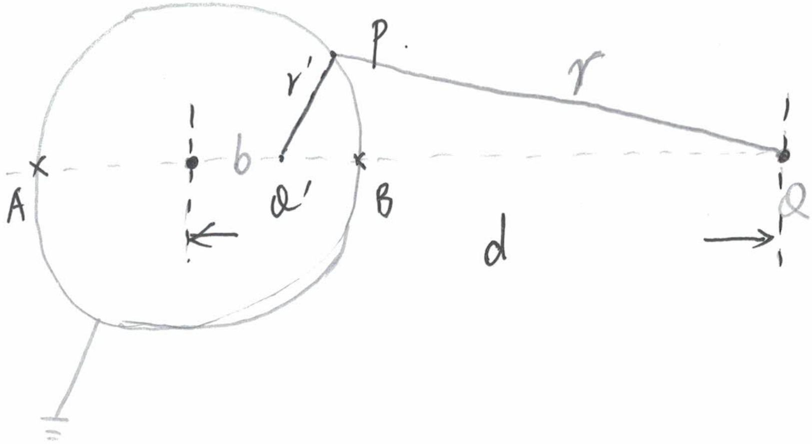
$\varphi_B = 2Q \ln \frac{R-b}{d-R}$

$\varphi_A = \varphi_B \Longrightarrow \frac{R+b}{R+d} = \frac{R-b}{d-R} \Longrightarrow \frac{R+b}{R-b} = \frac{d+R}{d-R}$

$\Longrightarrow \boxed{b = \frac{R^2}{d}}$ ✓

$2Q \rightarrow \frac{Q}{2\pi\varepsilon_0}$

柱形：镜像法可否使用?



$$\varphi = \frac{Q}{2\pi\varepsilon_0}\ln\left(\frac{a}{r}\right)$$ 选在有限远处 a

$$\varphi_P = \frac{Q'}{2\pi\varepsilon_0}\ln\left(\frac{a}{r'}\right) + \frac{Q}{2\pi\varepsilon_0}\ln\left(\frac{a}{r}\right)$$

A: $r' = R+b$, $r = d+R$

B: $r' = R-b$, $r = d-R$

$$\varphi_A = \frac{Q'}{2\pi\varepsilon_0}\ln\left(\frac{a}{R+b}\right) + \frac{Q}{2\pi\varepsilon_0}\ln\left(\frac{a}{d+R}\right)$$

$$\varphi_B = \frac{Q'}{2\pi\varepsilon_0}\ln\left(\frac{a}{R-b}\right) + \frac{Q}{2\pi\varepsilon_0}\ln\left(\frac{a}{d-R}\right)$$

不确定能否用镜像法?

选柱中心为电势零点

$$\varphi_{Q'} = \frac{Q'}{2\pi\varepsilon_0}\ln\left(\frac{b}{r'}\right)$$

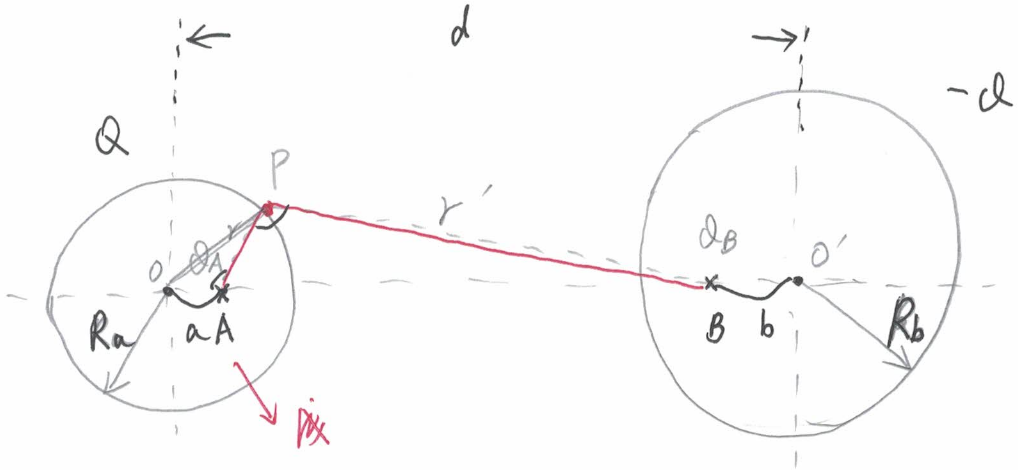
$$\varphi_Q = \frac{Q}{2\pi\varepsilon_0}\ln\left(\frac{d}{r}\right)$$

$$\varphi_A = \frac{\lambda'}{2\pi\varepsilon_0} \ln\left(\frac{b}{R+b}\right) + \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{d}{d+R}\right) = 0$$

$$\varphi_B = \frac{\lambda'}{2\pi\varepsilon_0} \ln\left(\frac{b}{R-b}\right) + \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{d}{d-R}\right) = 0$$

$$\varphi_P = \frac{\lambda'}{2\pi\varepsilon_0} \ln\left(\frac{b}{r'}\right) + \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{d}{r}\right)$$

电荷分布
电容等



电荷如何分布，电容为多少？

$$a = \frac{R_a^2}{OB}, \quad b = \frac{R_b^2}{O'A}$$

$$\rightsquigarrow \frac{OA}{R_a} = \frac{R_a}{OB}, \quad \frac{O'B}{R_b} = \frac{R_b}{O'A}$$

$$\rightsquigarrow \triangle OAP \backsim \triangle OPB$$

$$\frac{AP}{PB} = \frac{r}{r'} = \frac{OA}{OP} = \frac{a}{R_a}$$

入木三分

在a上：$\frac{R_a}{OB} = \frac{r}{r'} = \frac{R_a}{d-b} = \frac{a}{R_a}$

同样在b上：

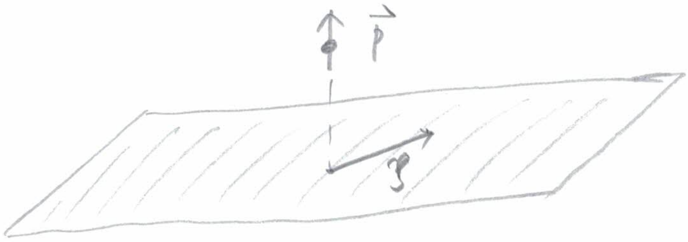
$$\frac{r}{r'} = \frac{R_b}{d-a} = \frac{b}{R_b}$$

圆柱导体的电势

$$\varphi_a = 2Q \ln\frac{L}{r} - 2Q \ln\frac{L}{r'} = -2Q \ln\frac{r}{r'} = -2Q \ln\frac{a}{R_b}$$

介电体与导体相互作用

$$\vec{p}\,\delta(x)\,\delta(y)\,\delta(z-a)$$



无导体时，$\vec{p}$ 产生的电场

$$\varphi(\vec{r}) = -\frac{1}{4\pi\varepsilon_0}\int_{T\to\infty}\frac{\nabla\cdot(\vec{p}\,\delta(x')\,\delta(y')\,\delta(z'-a))}{|\vec{r}-\vec{r}'|}\,d\vec{r}'$$

$$= -\frac{1}{4\pi\varepsilon_0}\,\partial_\beta\int_{T\to\infty}\frac{p_\beta\,\delta(x')\,\delta(y')\,\delta(z'-a)}{|\vec{r}-\vec{r}'|}\,d\vec{r}'$$

$$= -\frac{1}{4\pi\varepsilon_0}\,\partial_\beta\int_{T\to\infty}\frac{p_\beta\,\delta(x')\,\delta(y')\,\delta(z'-a)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\,d\vec{r}'$$

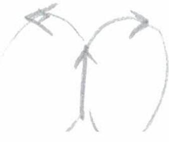
$$= -\frac{1}{4\pi\varepsilon_0}\,\partial_\beta\frac{p_\beta}{\sqrt{x^2+y^2+(z-a)^2}}$$

① $\vec{p}$ 极化沿 z

$$= -\frac{1}{4\pi\varepsilon_0}\,\partial_z\frac{p_z}{\sqrt{x^2+y^2+(z-a)^2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{p_z}{2}\,\frac{2(z-a)}{(x^2+y^2+(z-a)^2)^{\frac{3}{2}}} = \frac{p_z}{4\pi\varepsilon_0}\,\frac{(z-a)}{(x^2+y^2+(z-a)^2)^{\frac{3}{2}}}$$

$$\vec{E}(\vec{r}) = -\nabla\varphi(\vec{r})$$



引进镜像 $\boxed{P_z}$，$-a$ 处

$$\varphi(\vec{r}) = \frac{P_z}{4\pi\varepsilon_0}\left(\frac{z-a}{(\rho^2+(z-a)^2)^{\frac{3}{2}}} + \frac{z+a}{(\rho^2+(z+a)^2)^{\frac{3}{2}}}\right)\Bigg|_{z=0\text{处}}$$

$$= \frac{P_z}{4\pi\varepsilon_0}\left(\frac{-a}{(\rho^2+a^2)^{\frac{3}{2}}} + \frac{a}{(\rho^2+a^2)^{\frac{3}{2}}}\right) = 0$$

②. P沿 y 方向

$$\varphi(\vec{r}) = -\frac{1}{4\pi\varepsilon_0}\partial_y \frac{P_y}{\sqrt{x^2+y^2+(z-a)^2}}$$

$$= -\frac{1}{4\pi\varepsilon_0}\left(-\frac{P_y}{2}\right)\frac{2y}{\left[x^2+y^2+(z-a)^2\right]^{\frac{3}{2}}}$$

$$= \frac{1}{4\pi\varepsilon_0}P_y\frac{y}{\left[x^2+y^2+(z-a)^2\right]^{\frac{3}{2}}}$$

引进镜像 $\boxed{-P_y}$，$-a$ 处

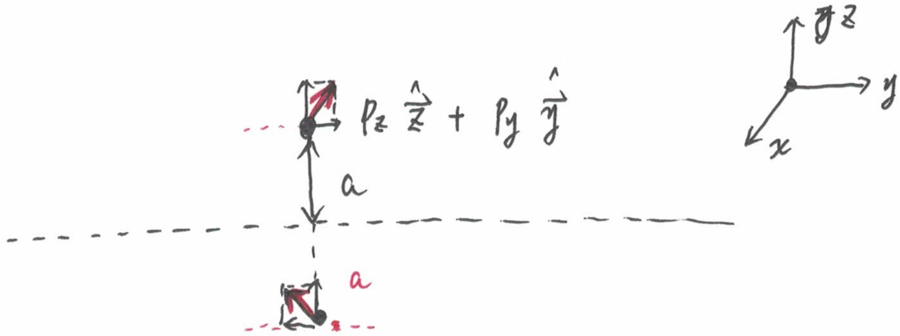
$$\varphi(\vec{r}) = \frac{P_y}{4\pi\varepsilon_0}\left(\frac{y}{(x^2+y^2+(z-a)^2)^{\frac{3}{2}}} - \frac{y}{(x^2+y^2+(z+a)^2)^{\frac{3}{2}}}\right)\Bigg|_{z=0}$$

$$= 0$$

吸引还是排斥

$\vec{p} = p_z\hat{z} + p_y\hat{y}$ ，处于 $z=a$ 处

镜像 $\vec{p} = p_z\hat{z} - p_y\hat{y}$，处于 $z=-a$ 处



$U_e = \frac{1}{2}\int \vec{E}(\vec{r})\cdot\vec{D}(\vec{r})\,d\vec{r}$ ， $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

$= \frac{\epsilon}{2}\int \vec{E}(\vec{r})\cdot\vec{E}(\vec{r})\,d\vec{r} + \frac{1}{2}\int \vec{E}(\vec{r})\cdot\vec{P}(\vec{r})\,d\vec{r}$

$$\varphi(\vec{r}) = \frac{p_z}{4\pi\epsilon_0}\left(\frac{z-a}{(x^2+y^2+(z-a)^2)^{\frac{3}{2}}} + \frac{z+a}{(x^2+y^2+(z+a)^2)^{\frac{3}{2}}}\right)$$

$$+ \frac{p_y}{4\pi\epsilon_0}\left(\frac{y}{(x^2+y^2+(z-a)^2)^{\frac{3}{2}}} - \frac{y}{(x^2+y^2+(z+a))^{\frac{3}{2}}}\right)$$

$\frac{1}{2}\int(-\nabla\varphi)\cdot\vec{p}(\vec{r})\,d\vec{r}$

$= \frac{1}{2}(-\nabla\varphi)\cdot\vec{p}(\vec{r})\Big|_{x=0,\ y=0,\ z=a}$

两个 dipole 的能量似乎该是镜像产生的电场与 $\vec{p}$ 的能量.

$$\varphi_{鏡}(\vec{r}) = \frac{P_z}{4\pi\varepsilon_0}\frac{z+a}{\left(x^2+y^2+(z+a)^2\right)^{\frac{3}{2}}} - \frac{P_y}{4\pi\varepsilon_0}\frac{y}{\left(x^2+y^2+(z+a)^2\right)^{\frac{3}{2}}}$$

$$-\int \nabla\varphi_{鏡}(\vec{r})\cdot\vec{P}(\vec{r})\,d\vec{r}$$

$\frac{1}{2}$ 元

$$= -\int \varphi_{鏡}(\vec{r})\,\nabla\cdot\vec{P}(\vec{r})\,d\vec{r}$$

$$= -\int \varphi_{鏡}(\vec{r})\left[P_y\frac{\partial\delta(y)}{\partial y}\delta(x)\,\delta(z-a) + P_z\frac{\partial\delta(z-a)}{\partial z}\delta(x)\,\delta(y)\right]dx\,dy\,dz$$

$$\nabla\varphi_{鏡}(\vec{r}) = \boxed{\frac{\partial\varphi_{鏡}(\vec{r})}{\partial x}}\,\hat{x} + \boxed{\frac{\partial\varphi_{鏡}(\vec{r})}{\partial y}}\,\hat{y} + \boxed{\frac{\partial\varphi_{鏡}(\vec{r})}{\partial z}}\,\hat{z}$$

✓ ✓

$$\frac{\partial\varphi_{鏡}(\vec{r})}{\partial y} = \frac{P_z}{4\pi\varepsilon_0}\frac{-(z+a)\frac{3}{2}\left(x^2+y^2+(z+a)^2\right)^{\frac{1}{2}}\cdot 2y}{\left(x^2+y^2+(z+a)^2\right)^3}$$

$$-\frac{P_y}{4\pi\varepsilon_0}\frac{\left(x^2+y^2+(z+a)^2\right)^{\frac{3}{2}} - y\frac{3}{2}\left(x^2+y^2+(z+a)^2\right)^{\frac{1}{2}}\cdot 2y}{\left(x^2+y^2+(z+a)^2\right)^3}$$

~~x=0, y=0, z=a~~

$$= \frac{P_z}{4\pi\varepsilon_0}\frac{-3y(z+a)}{\left(x^2+y^2+(z+a)^2\right)^{\frac{5}{2}}}$$

$$-\frac{P_y}{4\pi\varepsilon_0}\frac{x^2-2y^2+(z+a)^2}{\left(x^2+y^2+(z+a)^2\right)^{\frac{5}{2}}}$$

$x=0, y=0, z=a$

$$= -\frac{P_y}{4\pi\varepsilon_0}\frac{1}{8a^3}$$

$$\frac{\partial \varphi_{鏡}(\vec{r})}{\partial z} = \frac{p_z}{4\pi\varepsilon_0} \frac{(x^2+y^2+(z+a)^2)^{\frac{3}{2}} - (z+a)\frac{3}{2}(x^2+y^2+(z+a)^2)^{\frac{3}{2}} 2y}{(x^2+y^2+(z+a)^2)^3}$$

$$- \frac{p_y}{4\pi\varepsilon_0} \frac{-2(z+a)\frac{3}{2}(x^2+y^2+(z+a)^2)^{\frac{1}{2}}}{(x^2+y^2+(z+a)^2)^3}$$

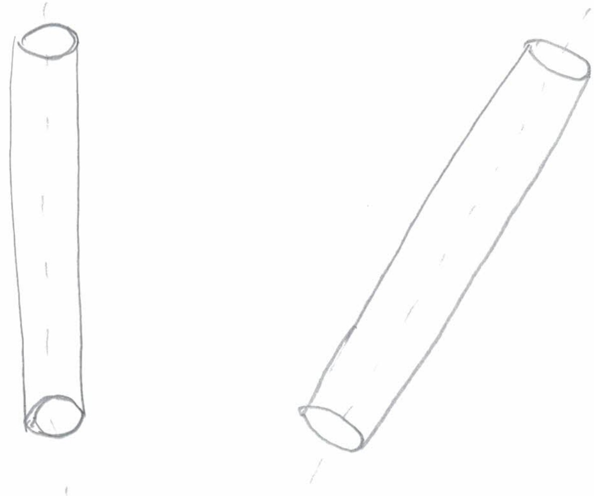
$$\overset{x=0, y=0, z=a}{=} \frac{p_z}{4\pi\varepsilon_0} \frac{1}{8a^3} + \frac{3p_y}{4\pi\varepsilon_0} \frac{1}{8a^3}$$

$$\boxed{U_{p\text{-}p} = -\int \nabla\varphi_{鏡}(\vec{r}) \cdot \vec{p}(\vec{r})\, d\vec{r}}$$

$$= -\frac{p_y^2}{4\pi\varepsilon_0} \frac{1}{8a^3} - \frac{p_z^2}{4\pi\varepsilon_0} \frac{1}{8a^3} - \frac{3p_y p_z}{4\pi\varepsilon_0} \frac{1}{8a^3} \quad (?)$$

答案は $-(2p_z^2 + p_y^2)/(8a^3)/4\pi\varepsilon_0$ 吸引

Current at steady state



一根导线中的电流分布

多根导线由于相互作用的电流分布

The steady motion of charges in conductors

Maxwell 方程组

$$\nabla \cdot \vec{E} = \frac{\rho_f(\vec{r})}{\varepsilon_0}$$

$$\nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}_f(\vec{r}) \Longrightarrow \boxed{\nabla \cdot \vec{j}_f = 0}$$ 电流无散度.

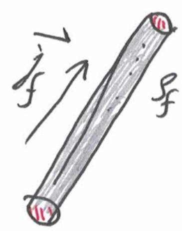
欧姆定律

$$\vec{j}_f = \sigma \vec{E}$$ （我们暂时只考虑这种简单情况）

$\Longrightarrow \nabla\cdot\vec{E}=0 \Longrightarrow$ $\boxed{\rho_f=0}$ 不可能有自由电荷!

$\Downarrow$

$\nabla^2\varphi=0$



导线外是空气，$j_n=0$, $E_n=0$

$\boxed{\text{似乎无方程限制 } \vec{E}}$

$\boxed{\vec{j}_f \text{ 受到磁场作用吗?}}$

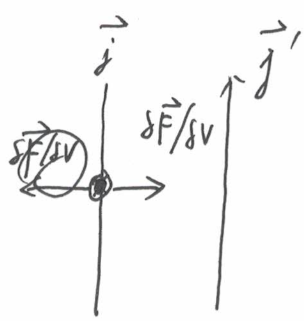
$$\vec{F}=q\vec{v}\times\vec{B}$$

$$q=\rho\,\Delta V$$

$$\Delta\vec{F}=\rho\,\Delta V\,\vec{v}(\vec{r})\times\vec{B}$$

$$\boxed{\vec{j}=\rho\vec{v}} \quad = \vec{j}(\vec{r})\times\vec{B}(\vec{r})\,\Delta V$$

$$\frac{\Delta\vec{F}}{\Delta V}=\vec{j}(\vec{r})\times\vec{B}(\vec{r})$$



单位体积元受到的力.

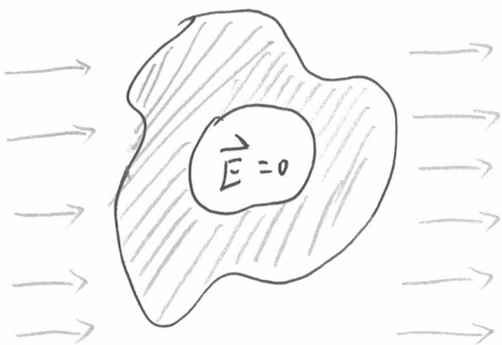
(后面讨论)

最后一招：严格求解边值问题

Jackson 书.

Magnetic shielding. → 磁屏蔽.

Spherical shell of permeable material in a uniform field.



导体 cavity 中
静电场为0

⟹ 原因是导体为等势体

问：磁场有没有类似效应？

导体： $\nabla\cdot\vec{E}=\dfrac{\rho_f}{\varepsilon_0}$ | $\nabla\cdot\vec{B}=0$

$\nabla\times\vec{E}=0$ | $\nabla\times\vec{H}=\vec{j}_f$

↳ 导体内.外 Laplace 方程 $\nabla^2\varphi=0$ + 边界条件

$\vec{B}=\mu_0\vec{H}+\mu_0\vec{M}\overset{\text{线性}}{=\!=}\mu\vec{H}$

⟹ $\nabla\cdot\vec{H}=0$, $\nabla\times\vec{H}=0$ ⟹ $\nabla^2\varphi=0$ Laplace 方程

$\vec{H}=-\nabla\varphi$

$\nabla^2\varphi=0$ + 边界条件 ⟹ 边值问题

(Boundary-value problem)

广义傅利叶级数 ⟹ 不同对称性下的基函数

球坐标：(r, θ, ϕ)

$$\boxed{\nabla^2 \psi = 0}$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0$$

$$\psi = \frac{U(r)}{r}\cdot P(\theta)\cdot Q(\phi)$$

$$\rightsquigarrow \frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2 \Longrightarrow Q = e^{\pm im\phi}$$，m 为整数

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0 \Longrightarrow U = Ar^{l+1} + Br^{-l}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0$$

↓ $x = \cos\theta$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0$$

↳ generalized Legendre equation

↳ solution: **associated Legendre functions**

$m=0$，轴对称

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+l(l+1)P=0$$

解 $P_l(x)$ 被称为勒让德多项式

$P_0(x)=1$
$P_1(x)=x$
$P_2(x)=\frac{1}{2}(3x^2-1)$
$P_3(x)=\frac{1}{2}(5x^3-3x)$
$P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$

⇝ 具有各种各样好的性质（学了一遍，之后一般是查询表格）

⇒ azimuthal symmetry 情形下的解我们就得到了

$$\Phi(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+B_l r^{-(l+1)}\right)P_l(\cos\theta)$$

由边界条件，我们就可定义 A_l, B_l 等，进而得到电势、电场、电荷分布等。

$m\neq0$ 时 (azimuthal variation)

generalized Legendre equation 的解为连带勒让德函数

$P_l^m(x)$

在这种情况下更常用的是球谐函数

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}$$

$m = -l, -(l-1), \cdots, 0, \cdots, (l-1), l$

$l=0$, $\quad Y_{00} = \frac{1}{\sqrt{4\pi}}$

$l=1$, $\quad \begin{cases} Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} \\ Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \end{cases}$

$\vdots$

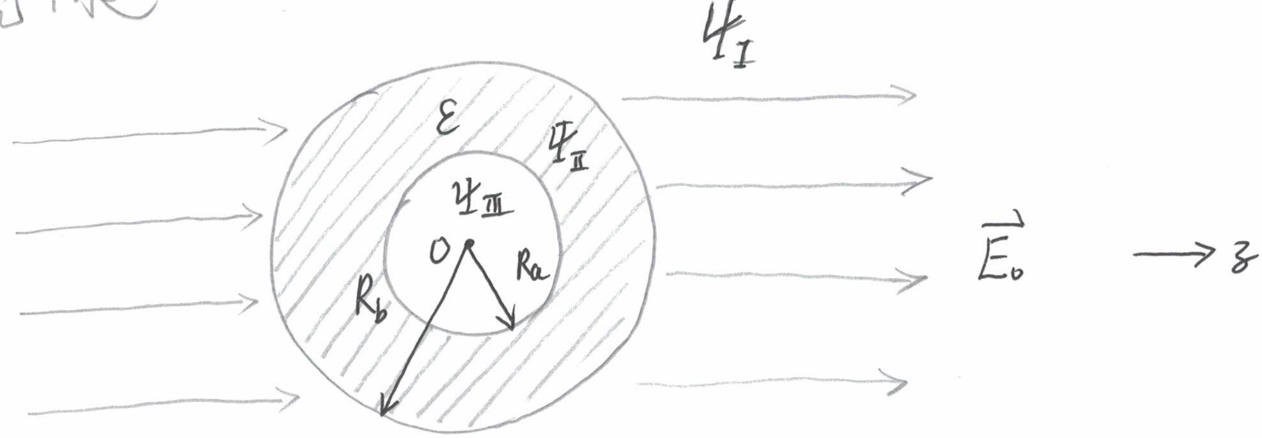
柱坐标系 (ρ, ϕ, z)

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$u(\rho,\phi,z) = R(\rho)\,Q(\phi)\,Z(z)$$

$\Longrightarrow$ 分离变量

介质球壳问题



介质球

取球心为球坐标的极点，取 $\vec{E_0}$ 方向为z轴.

显然这是一个球对称问题。电势零点取在球心O处

电势 φ_{I}，φ_{II}，φ_{III} 被分割到三个区域中

它们遵从 Laplace 方程 $\nabla^2\varphi=0$

因此具有一般性的解

$$\varphi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta)$$

$$\varphi_{I}(r\to\infty,\theta) = -E_0 z = -E_0 r\cos\theta$$

$\Rightarrow$ $A_1 = -E_0$，$A_{l\neq 1} = 0$，B_l 无法确定

$$\boxed{\varphi_{I}(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)}$$

$$\varphi_{III}(r\to 0,\theta) = 0$$

$\Rightarrow$ $B_l = 0$，$A_0 = 0$ $\Rightarrow$ $\boxed{\varphi_{III}(r,\theta) = \sum_{l=1}^{\infty} F_l r^l P_l(\cos\theta)}$

而 $\varphi_{II}(r,\theta)$ 暂时无法确定系数. $\varphi_{II}(r,\theta) = \sum_{l=0}^{\infty}\left(C_l r^l + D_l r^{-(l+1)}\right)P_l(\cos\theta)$

边界条件 $\nabla\cdot\vec{D}=0$, $\vec{D}=\varepsilon\vec{E}$

$\nabla\times\vec{E}=0$

$\vec{D}$ 法向分量连续， 即 $\varepsilon\frac{\partial\psi}{\partial r}$ 在球面上连续

ψ 本身连续

$$\psi_I(r=R_b,\theta)=\Psi_{II}(r=R_b,\theta) \quad ①$$

$$\varepsilon_0\frac{\partial\Psi_I}{\partial r}\Big|_{r=R_b}=\varepsilon\frac{\partial\Psi_{II}}{\partial r}\Big|_{r=R_b} \quad ②$$

$$\Psi_{II}(r=R_a,\theta)=\Psi_{III}(r=R_a,\theta) \quad ③$$

$$\varepsilon\frac{\partial\Psi_{II}}{\partial r}\Big|_{r=R_a}=\varepsilon_0\frac{\partial\Psi_{III}}{\partial r}\Big|_{r=R_a} \quad ④$$

先讨论 $R_a=0$ 的情况 $\rightarrow$ $\Psi_{II}(r,\theta)=\sum_{l=1}^{\infty}C_l r^l P_l(\cos\theta)$

$$-E_0R_bP_1(\cos\theta)+\sum_{l=0}^{\infty}B_lR_b^{-(l+1)}P_l(\cos\theta)=\sum_{l=1}^{\infty}C_lR_b^{l}P_l(\cos\theta)$$

相关系数应该相等

$$-E_0R_b+B_1R_b^{-2}=C_1R_b$$

$$B_0=0 \qquad (a)$$

$$B_lR_b^{-(l+1)}=C_lR_b^{l}\ ,\quad l\geqslant 2$$

$$\frac{\partial \varphi_{I}(r)}{\partial r} = -E_0 \cos\theta - \sum_{l=0}^{\infty} (l+1) B_l \, r^{-(l+2)} P_l(\cos\theta)$$

$$\frac{\partial \varphi_{II}(r)}{\partial r} = \sum_{l=1}^{\infty} C_l \, l \, r^{l} P_l(\cos\theta)$$

$$\Longrightarrow \varepsilon \sum_{l=1}^{\infty} C_l \, l \, R_b^{\,l} P_l(\cos\theta) = -\varepsilon_0 E_0 P_1(\cos\theta) - \varepsilon_0 \sum_{l=0}^{\infty} (l+1) B_l R_b^{-(l+2)} P_l(\cos\theta)$$

$$\Longrightarrow \left.\begin{aligned} &\varepsilon C_1 R_b = -\varepsilon_0 E_0 - 2\varepsilon_0 B_1 R_b^{-3} \\ &B_0 = 0 \\ &\varepsilon C_l \, l \, R_b^{\,l} = -\varepsilon_0 (l+1) B_l R_b^{-(l+2)}, \quad l \geq 2 \end{aligned}\right| \quad (b)$$

由 (a) 组与 (b) 组方程，我们得到

$$B_0 = 0$$

$$\begin{cases} -E_0 + B_1 R_b^{-3} = C_1 \\ \varepsilon C_1 R_b = -\varepsilon_0 E_0 - 2\varepsilon_0 B_1 R_b^{-3} \end{cases} \Longrightarrow \begin{cases} B_1 = \dfrac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 R_b^3 \\ C_1 = -E_0 + \dfrac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \\ \quad = -\dfrac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \end{cases}$$

$$\begin{cases} B_l R_b^{-(l+1)} = C_l R_b^{\,l} \\ -\varepsilon_0 (l+1) B_l R_b^{-(l+2)} = \varepsilon C_l \, l \, R_b^{\,l} \end{cases}, \quad l \geq 2$$

$$\hookrightarrow C_l, B_l = 0$$

$$\Longrightarrow \varphi_{I}(r,\theta) = -E_0 r \cos\theta + \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \frac{R_b^3}{r^2} \cos\theta$$

削弱 ↑

$$\varphi_{II}(r,\theta) = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \, r \, P_1(\cos\theta) \longrightarrow E_{内} = \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0$$

$R_b \neq 0$ 的情况，作业

现在将介质球壳换成导体球壳

⟹ 边界条件发生改变

①. φ 在球面上连续

②. φ 在球面上为等势体

$R_b \neq 0$ 时，作业

$R_a = 0$ 时，直接可以得出球体电势为0。

$$\varphi_I(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

$$\varphi_{II}(r,\theta) = \sum_{l=1}^{\infty} C_l r^l P_l(\cos\theta)$$

边界条件

$$-E_0 R_b \cos\theta + \sum_{l=0}^{\infty} B_l R_b^{-(l+1)} P_l(\cos\theta) = \sum_{l=1}^{\infty} C_l R_b^{l} P_l(\cos\theta)$$

$= 0$

⟹ $C_l = 0$，$l \geq 1$ ⟹ $-E_0 R_b + B_1 R_b^{-2} = 0$，$B_{l\geq 2} = 0$，$B_0 = 0$

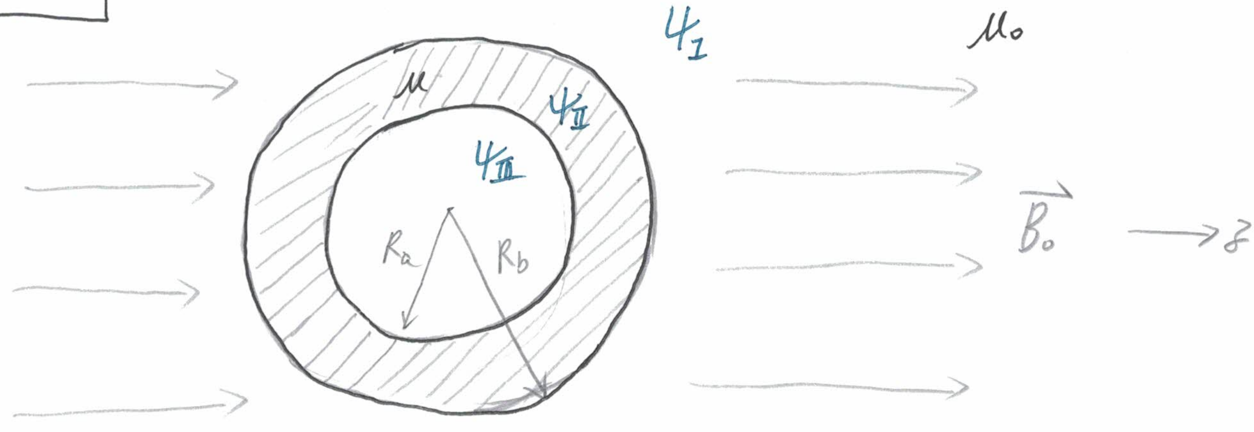
⟹ $B_1 = E_0 R_b^3$

⟹ $\varphi_I(r,\theta) = -E_0 r\cos\theta + E_0 \dfrac{R_b^3}{r^2} P_1(\cos\theta)$

$$\vec{E} = -\nabla\varphi_I(r,\theta) = E_0\left(1+\frac{2R_b^3}{r^3}\right)\cos\theta\,\vec{e}_\theta + E_0\left(-1+\frac{R_b^3}{r^3}\right)\sin\theta\,\vec{e}_\theta$$

$$\left(\nabla f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\vec{e}_\phi\right)$$

磁屏蔽



$\vec{B} = \mu \vec{H}$

$\nabla \cdot \vec{B} = 0 \Longrightarrow \nabla \cdot \vec{H} = 0$

$\nabla \times \vec{H} = 0$

$\vec{H} = -\nabla \varphi$

$\nabla^2 \varphi = 0$，除边界外均满足 Laplace 方程

$$\varphi(r,\theta) = \sum_{l} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos\theta)$$

在球外：

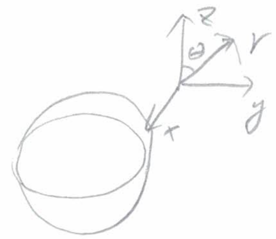
$$\varphi_I(r,\theta) = -H_0 r P_1(\cos\theta) + \sum_{l=0}^{\infty} \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta)$$

也可得到

$$\varphi_{III}(r,\theta) = \sum_{l=1}^{\infty} \delta_l r^l P_l(\cos\theta)$$

$$\varphi_{II}(r,\theta) = \sum_{l=0}^{\infty} \left(\beta_l r^l + \gamma_l \frac{1}{r^{l+1}} \right) P_l(\cos\theta)$$

边界条件:



① $\nabla\cdot\vec{B}=0 \Longrightarrow B_\perp$ 连续 $\Longrightarrow B_r$ 连续

② $\nabla\times\vec{H}=0 \Longrightarrow \vec{H}_\parallel$ 连续 $\Longrightarrow H_\theta$ 连续

$\Longrightarrow$ $\vec{H}$ 在不同区域的值算出，再拼接起来

$$\vec{H}_I = \left(-H_0 P_1(\cos\theta) - \sum_{l=0}^{\infty}(l+1)\frac{\alpha_l}{r^{l+2}}P_l(\cos\theta)\right)\vec{e}_r$$

$$+\left(-H_0 P_1'(\cos\theta) + \sum_{l=0}^{\infty}\frac{\alpha_l}{r^{l+2}}P_l'(\cos\theta)\right)\vec{e}_\theta$$

$$\vec{H}_{III} = \sum_{l=1}^{\infty} l\delta_l r^{l-1}P_l(\cos\theta)\,\vec{e}_r + \sum_{l=1}^{\infty}\delta_l r^{l-1}P_l'(\cos\theta)\,\vec{e}_\theta$$

$$\vec{H}_{II} = \sum_{l=0}^{\infty}\left(l\beta_l r^{l-1} - (l+1)\gamma_l\frac{1}{r^{l+2}}\right)P_l(\cos\theta)\,\vec{e}_r$$

$$+\sum_{l=0}^{\infty}\left(\beta_l r^{l-1} + \gamma_l\frac{1}{r^{l+2}}\right)P_l'(\cos\theta)\,\vec{e}_\theta$$

All efficients with $l\neq 1$ vanishes （关键）!

$$\mu_0\left(-H_0\cos\theta + \frac{\alpha_1}{R_b^3}\cos\theta\right) = \mu\left(\beta_1 R_b - 2\delta_1\frac{1}{R_b^3}\right)\cos\theta$$

$$\mu_0(\delta_1\cos\theta) = \mu\left(\beta_1 R_a - 2\gamma_1\frac{1}{R_a^3}\right)\cos\theta$$

$$H_0\sin\theta - \frac{\alpha_1}{R_b^3}\sin\theta = -\left(\beta_1 R_b + \delta_1\frac{1}{R_b^3}\right)\sin\theta \quad ✓$$

$$-\delta_1 R_a\sin\theta = -\left(\beta_1 R_a + \gamma_1\frac{1}{R_a^3}\right)\sin\theta \quad ✓$$

即：
$$\begin{cases} -H_0 + \dfrac{\alpha_1}{R_b^3} = \tilde{\mu}\left(\beta_1 - 2\delta_1 \dfrac{1}{R_b^3}\right) & ① \\ \delta_1 = \tilde{\mu}\left(\beta_1 R_a - 2\delta_1 \dfrac{1}{R_a^3}\right) \checkmark & ② \\ H_0 - \dfrac{\alpha_1}{R_b^3} = -\beta_1 - \dfrac{\delta_1}{R_b^3} \checkmark & ③ \\ -\delta_1 = -\beta_1 R_a - \dfrac{\delta_1}{R_a^3} \checkmark & ④ \end{cases}$$

$$\tilde{\mu} = \frac{\mu}{\mu_0}$$

~~由② ⇒ $\delta_1 = \dfrac{\tilde{\mu}\beta_1 R_a}{1 + \dfrac{2\tilde{\mu}}{R_a^3}}$~~

$$\alpha_1 = \frac{(2\tilde{\mu}+1)(\tilde{\mu}-1)}{(2\tilde{\mu}+1)(\tilde{\mu}+2) - \dfrac{R_a^3}{R_b^3}(\tilde{\mu}-1)^2}\left(R_b^3 - R_a^3\right) H_0$$

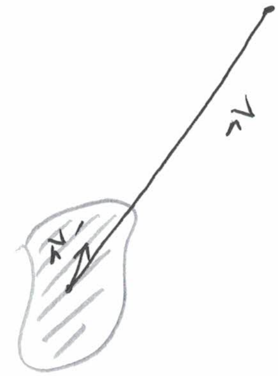
$$\delta_1 = -\frac{9\tilde{\mu}}{(2\tilde{\mu}+1)(\tilde{\mu}+2) - 2\dfrac{R_a^3}{R_b^3}(\tilde{\mu}-1)^2} H_0$$

当 $\tilde{\mu} \gg 1$ 时

$$\alpha_1 \longrightarrow \frac{2\tilde{\mu}^2}{2\tilde{\mu}^2 - \left(\dfrac{R_a}{R_b}\right)^3 \tilde{\mu}^2}\left(R_b^3 - R_a^3\right) H_0 = \frac{2}{2 - \left(\dfrac{R_a}{R_b}\right)^3}\left(R_b^3 - R_a^3\right) H_0$$

$$\delta_1 = -\frac{9\tilde{\mu}}{2\tilde{\mu}^2 - 2\left(\dfrac{R_a}{R_b}\right)^3 \tilde{\mu}^2} H_0 = -\frac{9 H_0}{2\tilde{\mu} - 2\left(\dfrac{R_a}{R_b}\right)^3 \tilde{\mu}} \otimes$$

多极矩展开（电多极矩，磁多极矩）



给定电荷分布 → 确定电势，电场

给定电流分布 → 确定矢势，磁场

前面已给给出了一般解：

$$\varphi(\vec{r}) = \int_V \frac{\rho(\vec{r}')\,dV'}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|}$$

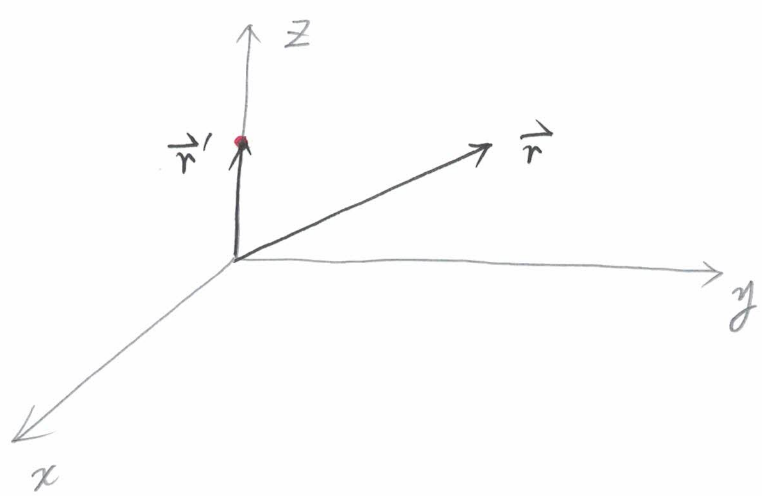
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}')\,dV'}{|\vec{r}-\vec{r}'|}$$

① 我们给出特殊形状的严格解

② 我们利用 Weyl identity 分析进近场问题

③ 有些情况下，关于的区域离源较远，此时允许由用近似方法处理各种问题 ⟹ 多极矩展开.

关键：$\dfrac{1}{|\vec{r}-\vec{r}'|}$ ⟹ 电动力学始终要处理这个！



$$\frac{1}{|\vec{r}-\vec{r}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi)$$

$r_>$ 为 r 与 r' 中大的

$r_<$ 为 r 与 r' 中小的

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d\vec{r}'$$

$$= \frac{1}{\varepsilon_0} \int \rho(\vec{r}') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi) d\vec{r}'$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{(2l+1)\varepsilon_0} \left[\int \rho(\vec{r}') \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') d\vec{r}' \right] Y_{lm}(\theta,\phi)$$

($r_< = r'$, $r > r'$, 因我们对源外电势感兴趣

$$= \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \underbrace{\left[\int \rho(\vec{r}') r'^l Y_{lm}^*(\theta',\phi') d\vec{r}' \right]}_{q_{lm}} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

$$= \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \boxed{q_{lm}} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

q_{lm} → 多极矩.

r^{l+1} → decay 的速度越来越快!

利用球谐函数

计算多极矩非常方便, 容易将所有项写出

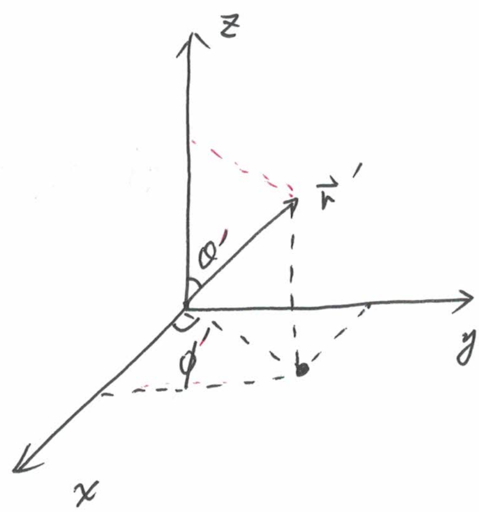
$l=0. \quad Y_{00} = \frac{1}{\sqrt{4\pi}}$

$l=1 \quad \begin{cases} Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi} \\ Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \end{cases}$

$l=2 \quad \begin{cases} Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta \, e^{2i\phi} \\ Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{i\phi} \\ Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \end{cases}$

...

$q_{00} = \frac{1}{\sqrt{4\pi}} \int \rho(\vec{r}\,') \, d\vec{r}\,' = \frac{1}{\sqrt{4\pi}} q$



$q_{11} = -\sqrt{\frac{3}{8\pi}} \int \rho(\vec{r}\,') \underbrace{r' \sin\theta' e^{-i\phi'}}_{x' \mp i y'} d\vec{r}\,'$

$= -\sqrt{\frac{3}{8\pi}} \int \rho(\vec{r}\,') (x' - iy') \, d\vec{r}\,'$

$= -\sqrt{\frac{3}{8\pi}} (p_x - i p_y)$

$q_{10} = \sqrt{\frac{3}{4\pi}} p_z$

$\frac{1}{12}\sqrt{\frac{15}{2\pi}} (Q_{11} - 2i Q_{12} - Q_{22})$

$\|$

$q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int \rho(\vec{r}\,') \left(\sin\theta' e^{-i\phi'}\right)^2 d\vec{r}\,'$

$= \frac{1}{4}\sqrt{\frac{15}{...}} \int \rho(\vec{r}\,') (x' - iy')^2 d\vec{r}\,' = \frac{1}{12}\sqrt{\frac{15}{2\pi}} \int 3\rho(\vec{r}\,')(x'x' - 2i x'y' - y'y') d$

计算电多极矩的例.

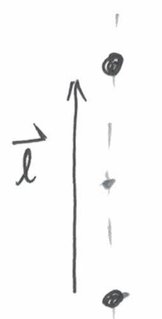
①. $Q = \int \rho(\vec{r}\,')\, d\vec{r}\,'$

$$\vec{p} = \int \vec{r}\,' \rho(\vec{r}\,')\, d\vec{r}\,'$$

$$Q_{ij} = \int \left(3x_i' x_j' - r'^2 \delta_{ij}\right) \rho(\vec{r}\,')\, d\vec{r}\,'$$

②. 若一个体系电荷分布对原点对称，它的电偶极矩为0.

⟹ **电偶极矩衡量电荷分布对原点的对称状况**



$Q \Rightarrow Q\delta(x)\delta(y)\delta(z-z_+)$

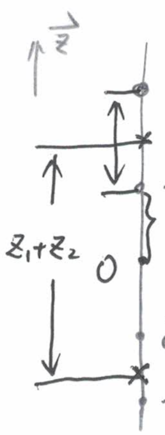
$-Q \Rightarrow -Q\delta(x)\delta(y)\delta(z-z_-)$

$$\vec{p} = Q\int \vec{r}\,' \delta(x')\delta(y')\delta(z'-z_+)\, d\vec{r}\,' - Q\int \vec{r}\,' \delta(x')\delta(y')\delta(z'-z_-)\, d\vec{r}\,'$$

$\vec{r}\,' \to (x'\hat{x} + y'\hat{y} + z'\hat{z})$

$$= Q z_+ \hat{z} - Q z_- \hat{z} = Q\underbrace{(z_+ - z_-)\hat{z}}_{\vec{l}} = Q\vec{l}$$

③. 电四极矩可看作"电偶极矩"的对



$+Q \Rightarrow +Q\delta(x)\delta(y)\delta(z-z_1)$

$-Q \Rightarrow -Q\delta(x)\delta(y)\delta(z-z_2)$

$+Q \Rightarrow +Q\delta(x)\delta(y)\delta(z-z_3)$

$-Q \Rightarrow -Q\delta(x)\delta(y)\delta(z-z_4)$

$$Q_{ij} = Q\int (3x_i' x_j' - r'^2\delta_{ij})\,\delta(x')\,\delta(y')\,\delta(z'-z_1)\,d\vec{r}'$$

$$- Q\int (3x_i' x_j' - r'^2\delta_{ij})\,\delta(x')\,\delta(y')\,\delta(z'-z_2)\,d\vec{r}'$$

$$\mp Q\int (3x_i' x_j' - r'^2\delta_{ij})\,\delta(x')\,\delta(y')\,\delta(z'-z_3)\,d\vec{r}'$$

$$+ Q\int (3x_i' x_j' - r'^2\delta_{ij})\,\delta(x')\,\delta(y')\,\delta(z'-z_4)\,d\vec{r}'$$

$$\Longrightarrow Q_{xx} = Q\int (3x'^2 - r'^2)\,\delta(x')\,\delta(y')\,\delta(z'-z_1)\,d\vec{r}'$$

$$- Q\int (3x'^2 - r'^2)\,\delta(x')\,\delta(y')\,\delta(z'-z_2)\,d\vec{r}'$$

$$\mp Q\int (3x'^2 - r'^2)\,\delta(x')\,\delta(y')\,\delta(z'-z_3)\,d\vec{r}'$$

$$+ Q\int (3x'^2 - r'^2)\,\delta(x')\,\delta(y')\,\delta(z'-z_4)\,d\vec{r}'$$

$$= Q(-z_1^2) - Q(-z_2^2) + Q(-z_3^2) - Q(-z_4^2)$$

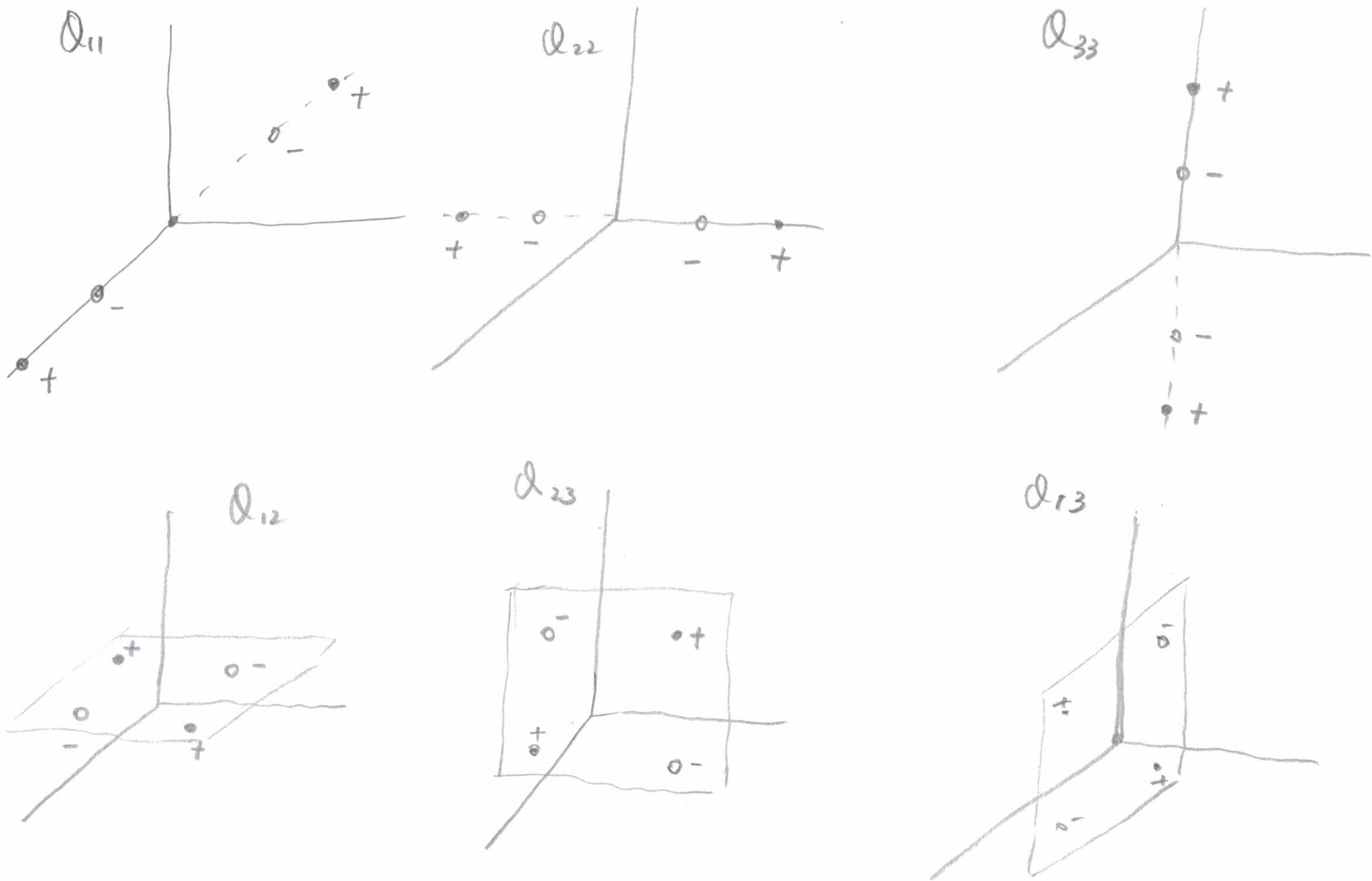
若选择合适的坐标原点，此项为0.

$$Q_{zz} = Q\,3z_1^2 - Q\,3z_2^2 \mp Q\,3z_3^2 + Q\,3z_4^2$$

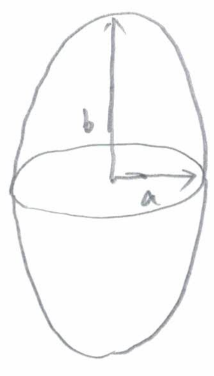
$$= 6Q(z_1^2 - z_2^2) = 6Q(z_1+z_2)(z_1-z_2) = 6pl$$

(z_1+z_2) ⇓ 两个电偶极子之间的矩离 l

$Q(z_1-z_2)$ ⇓ p : $p = Q(z_1 - z_2)$



作业：计算均匀带电椭球(半短轴 a，半长轴 b，电荷密度 ρ_0)的电四极矩。



$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}\le 1$$

利用这种方式可将多极矩计算完整刻成表格.

~~$\vec{?} = \int \vec{x}' \rho(\vec{x}') d$~~

$\vec{p} = \int \vec{r}' \rho(\vec{r}') d\vec{r}'$ ⟹ electric dipole moment

$Q_{ij} = \int (3x_i' x_j' - r'^2 \delta_{ij}) \rho(\vec{r}') d\vec{r}'$ ⟹ 电四极矩.

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \cdots\right)$$

原子核物理中，原子核可发生形变，电四极矩成为研究这一形变的重要物理量.

矢势的多极展开

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} d\vec{r}'$$

$$= \mu_0 \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\underbrace{\left[\int \vec{J}(\vec{r}') r'^l Y_{lm}^*(\theta',\phi') d\vec{r}'\right]}_{\vec{q}_{lm}} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

$\vec{q}_{00} = \frac{1}{\sqrt{4\pi}}\int \vec{J}(\vec{r}') d\vec{r}' = 0$ ⟹ 稳恒电流. ⟹ $\vec{A}^{(0)} = 0$

(作业!)

$\vec{q}_{11} = -\sqrt{\frac{3}{8\pi}}\int \vec{J}(\vec{r}')(x' - iy') d\vec{r}'$, $\vec{q}_{10} = \sqrt{\frac{3}{4\pi}}\int \vec{J}(\vec{r}') z' d\vec{r}'$

$$\vec{A}^{(1)} = \frac{\mu_0}{3}\vec{q}_{11}\frac{Y_{11}(\theta,\phi)}{r^2} + \frac{\mu_0}{3}\vec{q}_{10}\frac{Y_{10}(\theta,\phi)}{r^2}$$

$$Y_{l-m}(\theta,\phi) = (-1)^m Y_{lm}^*(\theta,\phi)$$

$$= \frac{\mu_0}{3}\frac{3}{8\pi}\frac{(x+iy)}{r^3}\int \vec{J}(\vec{r}')(x'-iy')\,d\vec{r}'$$

$$+ \frac{\mu_0}{3}\frac{3}{4\pi}\frac{z}{r^3}\int \vec{J}(\vec{r}')\,z'\,d\vec{r}'$$

$$= \frac{\mu_0}{8\pi}\frac{(x+iy)}{r^3}\int \vec{J}(\vec{r}')(x'-iy')\,d\vec{r}' + H.C.$$

$$+ \frac{\mu_0}{4\pi}\frac{z}{r^3}\int \vec{J}(\vec{r}')\,z'\,d\vec{r}'$$

$(x+iy)(x'-iy')$

$(x-iy)(x'+iy')$

$= xx' - ixy' - iyx' + yy'$

$+ xx' + ixy' - iyx' + yy'$

$= 2(xx'+yy')$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\cdot\int \vec{J}(\vec{r}')(\vec{r}'\cdot\vec{r})\,d\vec{r}'$$

what is this ?

$$\vec{r}\cdot\int \vec{r}'\,J_i\,d\vec{r}'$$

$$= \sum_j x_j \int x_j'\,J_i\,d\vec{r}' = ?$$

电流的矩如何算？

首先证明 $\int (x_i' J_j + x_j' J_i)\,d\vec{r}' = 0$

$$\int \nabla'\cdot(fg\vec{J})\,d\vec{r}' = 0, \text{ if } \vec{J} \text{ is localized}$$

$$= \int [\nabla'(fg)\cdot\vec{J} + fg\,\nabla'\cdot\vec{J}]\,d\vec{r}'$$

$$= \int [g\nabla' f\cdot\vec{J} + f\nabla' g\cdot\vec{J} + fg\,\nabla'\cdot\vec{J}]\,d\vec{r}'$$

令 $f = x_i'$, $g = x_j'$, $\nabla'\cdot\vec{J} = 0$

$$\Longrightarrow \int [x_j' \nabla' x_i'\cdot\vec{J} + x_i' \nabla' x_j'\cdot\vec{J}]\,d\vec{r}' = 0$$

$$\Longrightarrow \boxed{\int (x_j' J_i + x_i' J_j)\,d\vec{r}' = 0}$$

$$\Longrightarrow \sum_j x_j \int x_j' J_i\,d\vec{r}'$$

$$= \frac{1}{2}\sum_j x_j \int x_j' J_i\,d\vec{r}' + \frac{1}{2}\sum_j x_j \underbrace{\int x_j' J_i\,d\vec{r}'}_{= -\int x_i' J_j\,d\vec{r}'}$$

$$= -\frac{1}{2}\sum_j x_j \int (x_i' J_j - x_j' J_i)\,d\vec{r}'$$

$$= -\frac{1}{2}\sum_{j,k} \epsilon_{ijk}\, x_j \int (\vec{r}'\times\vec{J})_k\,d\vec{r}'$$

$$= -\frac{1}{2}\left[\vec{r}\times\int(\vec{r}'\times\vec{J})\,d\vec{r}'\right]_i$$

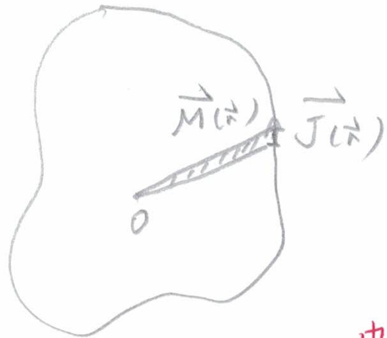
$$= \vec{r}\cdot\int \vec{r}' J_i\,d\vec{r}'$$

指标分量

$$\Longrightarrow \vec{A}^{(1)} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left(-\frac{1}{2}\right)\vec{r}\times\int(\vec{r}'\times\vec{J}(\vec{r}'))\,d\vec{r}'$$

定义 magnetic moment density

$$\vec{M}(\vec{r}) = \frac{1}{2}\,\vec{r}\times\vec{J}(\vec{r})$$

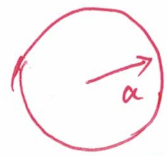


定义 magnetic moment

$$\vec{m} = \frac{1}{2}\int d\vec{r}'\,\vec{r}'\times\vec{J}(\vec{r}')$$

$$\Longrightarrow\quad \vec{A}^{(1)} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\,\vec{m}\times\vec{r}$$

电流环



$I\,\delta(r-a)\,\delta(z)$

$$\frac{I}{2}\int d\vec{r}'\, r'\,\delta(r'-a)\,\delta(z')$$

$$= \frac{I}{2}\int r'dr'\,d\varphi'\,dz'\,r'\,\delta(\ldots)$$

$$= \frac{I}{2}\,2\pi a^2 = I\pi a^2$$

最后，计算磁场

$$\vec{B}^{(1)} = \nabla\times\vec{A}^{(1)}$$

$$= \frac{\mu_0}{4\pi}\nabla\times\left(\frac{1}{r^3}\,\vec{m}\times\vec{r}\right)$$

$$B_k^{(1)} = \frac{\mu_0}{4\pi}\varepsilon_{ijk}\frac{\partial}{\partial x_i}\left(\frac{1}{r^3}\,\vec{m}\times\vec{r}\right)_j$$

$$= \frac{\mu_0}{4\pi}\varepsilon_{ijk}\frac{\partial}{\partial x_i}\left(\varepsilon_{i'j'j}\frac{1}{r^3}m_{i'}x_{j'}\right)$$

$$= \frac{\mu_0}{4\pi}\underbrace{\varepsilon_{ijk}\varepsilon_{i'j'j}}_{=-\varepsilon_{ikj}\varepsilon_{i'j'j}}\frac{\partial}{\partial x_i}\left(\frac{1}{r^3}m_{i'}x_{j'}\right)$$

关于全反对称张量进行指标计算参见下页

$$= -\frac{\mu_0}{4\pi}\left(\delta_{ii'}\delta_{kj'} - \delta_{ij'}\delta_{ki'}\right)\frac{\partial}{\partial x_i}\left(\frac{1}{r^3}m_{i'}x_{j'}\right)$$

$$= -\frac{\mu_0}{4\pi}\left[\frac{\partial}{\partial x_i}\left(\frac{1}{r^3} m_i x_k\right) - \frac{\partial}{\partial x_i}\left(\frac{1}{r^3} m_k x_i\right)\right]$$

$$= \frac{\mu_0}{4\pi}\frac{\partial}{\partial x_i}\left(\frac{m_k x_i - m_i x_k}{r^3}\right)$$

$$= \frac{\mu_0}{4\pi}\frac{\frac{\partial}{\partial x_i}(m_k x_i - m_i x_k) r^3 - \left(\frac{\partial r^3}{\partial x_i}\right)(m_k x_i - m_i x_k)}{r^6}$$

$$= \frac{\mu_0}{4\pi}\frac{(3m_k - m_k) r^3 - 3r^2 \cdot \frac{1}{2}\frac{2x_i}{r}(m_k x_i - m_i x_k)}{r^6}$$

$\frac{\partial r}{\partial x_i}$

$\sqrt{x^2+y^2+z^2}$

$$= \frac{\mu_0}{4\pi}\frac{2m_k r^3 - 3r(m_k r^2 - x_i m_i x_k)}{r^6}$$

$$= \frac{\mu_0}{4\pi}\frac{-m_k r^2 + 3\vec{r}\cdot\vec{m}\, x_k}{r^5}$$

$$\Longrightarrow \vec{B}^{(1)} = \frac{\mu_0}{4\pi}\frac{3(\vec{r}\cdot\vec{m})\vec{r} - r^2\vec{m}}{r^5}$$

作业：与磁球所产生的磁场对比.

即推出最后结果

⟹ 熟练使用指标运算.

附:

利用 kronecker 符号 $\delta_{\alpha\beta}$, $\varepsilon_{\alpha\beta\gamma}$ 等进行指标运算.

（$\varepsilon_{\alpha\beta\gamma}$：全反对称张量）

$$C_\alpha = \varepsilon_{\alpha\beta\gamma} A_\beta B_\gamma$$

重复指标表示求和

$$C_x = \varepsilon_{xyz} A_y B_z + \varepsilon_{xzy} A_z B_y$$

$$= A_y B_z - A_z B_y$$ ，即上面推出的 x 分量.

全反对称张量 $\varepsilon_{\alpha\beta\gamma} = \begin{cases} 1, & \{x,y,z\} \\ -1, & \{x,z,y\} \end{cases}$

全反对称张量的乘积

$$\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha'\beta'\gamma'} = \begin{vmatrix} \delta_{\alpha\alpha'} & \delta_{\alpha\beta'} & \delta_{\alpha\gamma'} \\ \delta_{\beta\alpha'} & \delta_{\beta\beta'} & \delta_{\beta\gamma'} \\ \delta_{\gamma\alpha'} & \delta_{\gamma\beta'} & \delta_{\gamma\gamma'} \end{vmatrix}$$

缩并:

$$\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha\beta'\gamma'} = \begin{vmatrix} \delta_{\beta\beta'} & \delta_{\beta\gamma'} \\ \delta_{\gamma\beta'} & \delta_{\gamma\gamma'} \end{vmatrix} = \delta_{\beta\beta'}\delta_{\gamma\gamma'} - \delta_{\beta\gamma'}\delta_{\gamma\beta'}$$

$\vec{m}$ 的物理意义

考察一下静磁场能量.

$$W_m = \frac{1}{2}\int_{V\to\infty} \vec{B}\cdot\vec{H}\, d\vec{r}$$

$$= \frac{1}{2}\int_{V\to\infty} (\nabla\times\vec{A})\cdot\vec{H}\, d\vec{r}$$

$$\boxed{\nabla\cdot(\vec{A}\times\vec{H}) = (\nabla\times\vec{A})\cdot\vec{H} - \vec{A}\cdot(\nabla\times\vec{H})}$$

$$\frac{\partial}{\partial x_i}(\vec{A}\times\vec{H})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_i}(A_j H_k) = \varepsilon_{ijk}\frac{\partial A_j}{\partial x_i}H_k + \varepsilon_{ijk}A_j\frac{\partial H_k}{\partial x_i}$$

$$= (\nabla\times\vec{A})\cdot\vec{H} - (\nabla\times\vec{H})\cdot\vec{A}$$

$$= \frac{1}{2}\int_{V\to\infty}\Big[\nabla\cdot(\vec{A}\times\vec{H}) + \vec{A}\cdot\underbrace{(\nabla\times\vec{H})}_{\vec{J}_f}\Big]\, d\vec{r}$$

$$= \frac{1}{2}\int_{S_\infty}(\vec{A}\times\vec{H})\cdot d\vec{S} + \frac{1}{2}\int_{V\to\infty}\vec{J}_f\cdot\vec{A}\, d\vec{r}$$

$\vec{A}\propto\frac{1}{r}$, $\vec{H}\propto\frac{1}{r^2}$, $S\sim r^2$, 第一项积分趋于零

$$= \frac{1}{2}\int_{V\to\infty}\vec{J}_f(\vec{r})\,\vec{A}\, d\vec{r}$$

$$= \frac{1}{2}\int_{V\to V_{源}}\vec{J}_f(\vec{r})\cdot\vec{A}\, d\vec{r}$$

$\Longrightarrow$ 可利用电流密度及矢势 $\vec{A}$ 计算静磁场总能量.

考虑外磁场与 local 电流源的相互作用，计算它们的相互作用能

$$\vec{J}_f = \vec{J}_f^{(s)} + \vec{J}_f^{(e)}$$

$\vec{J}_f^{(s)}$ ↑ 感兴趣的电流源

$\vec{J}_f^{(e)}$ ↑ 产生外磁场的电流.

$$\vec{A} = \vec{A}^{(s)} + \vec{A}^{(e)}$$

$$W_m = \frac{1}{2}\int_{V\to\infty} \left(\vec{J}_f^{(s)} + \vec{J}_f^{(e)}\right)\cdot\left(\vec{A}^{(s)} + \vec{A}^{(e)}\right) d\vec{r}$$

$$= \frac{1}{2}\int_{V\to\infty} \vec{J}_f^{(s)}\cdot\vec{A}^{(s)}\, d\vec{r} \rightarrow \text{源自能}$$

$$+ \frac{1}{2}\int_{V\to\infty} \vec{J}_f^{(s)}\cdot\vec{A}^{(e)}\, d\vec{r}$$

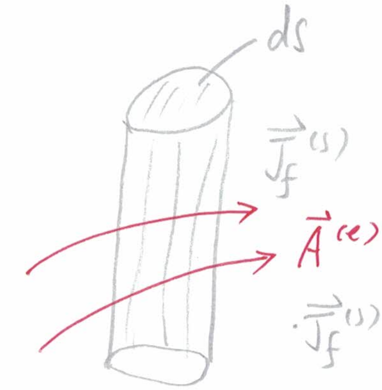
$$+ \frac{1}{2}\int_{V\to\infty} \vec{J}_f^{(e)}\cdot\vec{A}^{(s)}\, d\vec{r}$$

} 相互作用能，两部分是相等（证明：作业）

$$+ \frac{1}{2}\int_{V\to\infty} \vec{J}_f^{(e)}\cdot\vec{A}^{(e)}\, d\vec{r} \rightarrow \text{外磁场自能}$$

$$W_{in} = \int_{V_0} \vec{J}_f^{(s)}\cdot\vec{A}^{(e)}\, dV$$

$\vec{J}_f^{(s)}$ ⇓ 小区域电流密度



$\vec{J}_f^{(s)} ds\, dl = \vec{J}_f^{(s)} ds\, d\vec{l} \rightarrow I\, d\vec{l}$

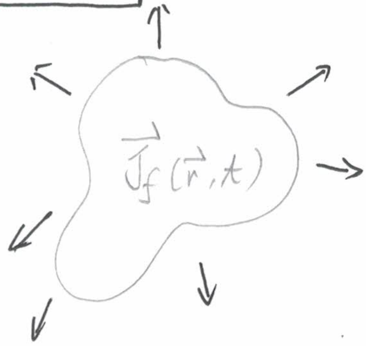
$$= I\oint_l \vec{A}^{(e)}\cdot d\vec{l}$$

Stokes 定理

$$= I\int_S \vec{B}^{(e)}\cdot d\vec{S} \approx \vec{B}(0)\, I\int d\vec{S} = \vec{B}(0)\cdot\vec{m}$$

（$\int_S \vec{B}^{(e)}\cdot d\vec{S}$：通量.）

电磁波的辐射



让电流含时，固定源问题

$\Downarrow$

场 ← Green function ⟷ 源

非常类似于静态问题

Maxwell 方程组

$$\begin{cases} \nabla\cdot\vec{D} = \rho_f \\ \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{H} = \vec{J}_f + \dfrac{\partial\vec{D}}{\partial t} \end{cases}$$

含时项均要考虑进来

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M})$$

$\Downarrow$ 简单情况

$\vec{D} = \varepsilon\vec{E}$, $\vec{B} = \mu\vec{H}$

+ 边界条件

磁矢势 $\vec{B} = \nabla\times\vec{A}$

$\to \nabla\times\vec{E} = -\nabla\times\dfrac{\partial\vec{A}}{\partial t} \Longrightarrow \nabla\times\left(\vec{E} + \dfrac{\partial\vec{A}}{\partial t}\right) = 0$

此时允许标势

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\nabla\varphi$$

$$\Longrightarrow \vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}$$

规范不变性: gauge invariance

$\vec{B}$, $\vec{E}$ 为物理量。$\vec{A}$, φ 为辅助场，其具有一定的自由度

$\vec{A} \to \vec{A} + \nabla\phi$， $\vec{B}$ 不变

要保持 $\vec{E}$ 不变，$\varphi \to \varphi - \frac{\partial\phi}{\partial t}$

我们称变换

$$\begin{cases} \vec{A} \to \vec{A}' = \vec{A} + \nabla\phi \\ \varphi \to \varphi' = \varphi - \frac{\partial\phi}{\partial t} \end{cases}$$ 为规范变换

由于同一物理量 $\{\vec{E}, \vec{B}\}$ 可对应很多组 $\{\vec{A}_{(i)}, \varphi_{(i)}\}$，不同组的 $\{\vec{A}_{(i)}, \varphi_{(i)}\}$ 之间由规范变换联系

那么为了方便地用 $\{\vec{A}, \varphi\}$ 求出 $\{\vec{E}, \vec{B}\}$，我们对其进行一定的限制，这种限制称为**规范固定**。

我们现在推导 $\{\vec{A}, \varphi\}$ 所满足的方程.

对于线性介质

$$\begin{cases} \nabla\cdot\vec{E} = \frac{\rho_f}{\varepsilon} & ① \\ \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \checkmark & ② \\ \nabla\cdot\vec{B} = 0 \checkmark & ③ \\ \nabla\times\vec{B} = \mu\vec{J_f} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t} & ④ \end{cases}$$

由④，$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \mu\vec{J_f} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}$ ⑦

$= \mu\vec{J_f} + \mu\varepsilon\frac{\partial}{\partial t}\left(-\frac{\partial\vec{A}}{\partial t} - \nabla\varphi\right) = \mu\vec{J_f} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} - \mu\varepsilon\frac{\partial}{\partial t}\nabla\varphi$

即：$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \mu\vec{J_f} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} - \mu\varepsilon\frac{\partial}{\partial t}\nabla\varphi$

$$\Longrightarrow \boxed{\nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} - \nabla\left(\nabla\cdot\vec{A} + \mu\varepsilon\frac{\partial\varphi}{\partial t}\right) = -\mu\vec{J_f}}$$

称为一般形式的达朗贝尔方程

由①可得

$$\nabla\cdot\left(-\nabla\varphi - \frac{\partial\vec{A}}{\partial t}\right) = \frac{\rho_f}{\varepsilon}$$

$$\Longrightarrow \boxed{\nabla^2\varphi + \frac{\partial}{\partial t}\nabla\cdot\vec{A} = -\frac{\rho_f}{\varepsilon}}$$

规范固定：

(1) 库仑规范 $\nabla\cdot\vec{A} = 0$ $\Longrightarrow$ $\vec{A}$无源场

$\nabla^2\varphi = -\frac{\rho_f}{\varepsilon}$ $\Longrightarrow$ ~~电场部分完全由~~ φ由ρ_f来求

优点：允许利用静电场的方法来讨论时变电场。

$\nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} - \nabla\left(\mu\varepsilon\frac{\partial\varphi}{\partial t}\right) = -\mu\vec{J_f}$

(2) 在时变电磁场问题中，更常取洛仑兹规范条件

$$\nabla\cdot\vec{A} + \mu\varepsilon\frac{\partial\varphi}{\partial t} = 0$$

$$\begin{cases}\nabla^2\varphi - \mu\varepsilon\dfrac{\partial^2\varphi}{\partial t^2} = -\dfrac{\rho_f}{\varepsilon}\\[2mm] \nabla^2\vec{A} - \mu\varepsilon\dfrac{\partial^2\vec{A}}{\partial t} = -\mu\vec{J_f}\end{cases}$$

两方程形式对称，给定源 $\rho_f \rightarrow \varphi$，$\vec{J_f} \rightarrow \vec{A}$，可分别求解 $\begin{cases}\vec{E}\\ \vec{B}\end{cases}$

问题：给定源问题，如何使用 Green function 方法求定场。

$$\oint_\Sigma \left(u \frac{dv}{dn} - v \frac{du}{dn} \right) dS = \int_T \left(u \nabla^2 v - v \nabla^2 u \right) dV$$

$$\nabla^2 \psi - \mu\varepsilon \frac{\partial^2 \psi}{\partial t^2} = -\frac{\rho_f(\vec{r}, t)}{\varepsilon}$$

假定源有固定频率 ω，$\rho_f(\vec{r}, t) = \rho_f(\vec{r}) e^{-i\omega t}$，

$$u(\vec{r}, t) = u(\vec{r}) e^{-i\omega t}$$

$$\nabla^2 \psi + \mu\varepsilon\omega^2 \psi = -\frac{\rho_f(\vec{r})}{\varepsilon}$$

$$\mu\varepsilon\omega^2 \equiv k^2$$

$\Longrightarrow$ $$\nabla^2 \psi(\vec{r}) + k^2 \psi(\vec{r}) = -\frac{\rho_f(\vec{r})}{\varepsilon}$$

$$\nabla^2 u(\vec{r}) + k^2 u(\vec{r}) = -\frac{\rho_f(\vec{r})}{\varepsilon}$$

$$\nabla^2 v(\vec{r}) + k^2 v(\vec{r}) = \delta(\vec{r} - \vec{r}_0)$$

$\Longrightarrow$ $$\vec{v}(\vec{r}) = -\frac{1}{4\pi} \frac{e^{ik|\vec{r} - \vec{r}_0|}}{|\vec{r} - \vec{r}_0|}$$ $\Longrightarrow$ 证明（作业）

右边 $= \int_T \left(u \nabla^2 v - v \nabla^2 u \right) dV$

$$= \int_T \left[u(\nabla^2 v + k^2 v) - v(\nabla^2 u + k^2 u) \right] dV$$

$$= u(\vec{r}_0) - \frac{1}{4\pi} \int_T \frac{\rho_f(\vec{r})}{\varepsilon} \frac{e^{ik|\vec{r} - \vec{r}_0|}}{|\vec{r} - \vec{r}_0|} dV = 左边$$

右边 $\oint_\Sigma \left(u\frac{dv}{dn} - v\frac{du}{dn} \right) dS$

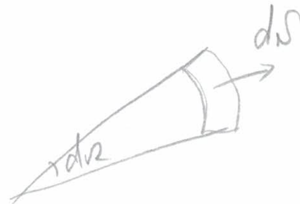
$$= -\frac{1}{4\pi}\oint_\Sigma \left(u\frac{d}{dn}\frac{e^{i|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} - \frac{e^{i|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|}\frac{du}{dn} \right) dS$$

(想象一大圆，半径R，圆心 $\vec{r}_0$

$\frac{d}{dn} = \frac{d}{dR}$

$$= -\frac{1}{4\pi}\oint_\Sigma \left(u\frac{d}{dR}\frac{e^{iR}}{R} - \frac{e^{iR}}{R}\frac{du}{dR} \right) dS$$

($\frac{\partial}{\partial R}\frac{e^{ikR}}{R} = \frac{e^{ikR}}{R^2}(ikR-1)$



$$= -\frac{1}{4\pi}\oint_\Sigma \left(u\frac{e^{ikR}}{R}ik - u\frac{e^{ikR}}{R^2} - \frac{e^{iR}}{R}\frac{du}{dR} \right) R^2 d\Omega$$

$$= -\frac{1}{4\pi}\oint_\Sigma \left(u e^{ikR} ikR - u e^{ikR} - R e^{iR}\frac{du}{dR} \right) d\Omega$$

$$= -\frac{1}{4\pi}\oint_\Sigma \left(u(R)\, ikR - u(R)\, R\frac{du}{dR} \right) e^{ikR} d\Omega$$

$R\to\infty$ 时，$u(R)\to 0$，要求该积分趋于0，则有条件

$$\oint_\Sigma \left(u(R)\, ikR - R\frac{du(R)}{dR} \right) e^{ikR} d\Omega = 0$$

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即辐射条件

由此得到

$$u(\vec{r}_0) = \frac{1}{4\pi\varepsilon}\int_T \rho_f(\vec{r})\,\frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|}\,d\vec{r}$$

$$\Rightarrow\quad \psi(\vec{r}) = \frac{1}{4\pi\varepsilon}\int_{V'} \rho_f(\vec{r}')\,\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\,d\vec{r}'$$

同样,

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\int_{V'} \vec{J}_f(\vec{r}')\,\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\,d\vec{r}'$$

可将时间因子 t 包括在源中.

作业: 验证 Lorentz 规范条件得到满足:

$$\nabla\cdot\vec{A} + \mu\varepsilon\frac{\partial\psi}{\partial t} = 0$$

物理意义:

① $\vec{r}$ 在 $\vec{r}'$ 外.

② $\vec{r}$ 在 $\vec{r}'$ 内 ?

$\frac{}{|\vec{r}-\vec{r}'|}$

剩下的工作

① Weyl 可求解的模型 线, 环, 球等

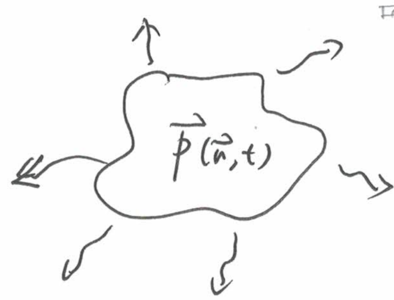
② Weyl identity, 近场

③ 多极矩展开, 远场

固定源间的

的数值统一的方法

探索: $\vec{P}$ 的变化, $\vec{M}$ 的变化 } 是否有辐射, 如何计算?



$\Rightarrow$

$$\nabla\cdot\vec{D} = 0$$
$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\nabla\cdot\vec{B} = 0$$
$$\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$$

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$\vec{M}(\vec{r},t) = \vec{M}(\vec{r}) + \vec{m}(\vec{r},t)$ —般为这种情况!

推迟势

$$\varphi(\vec{r},t)=\frac{1}{4\pi\varepsilon}\int_{V'}\rho_f(\vec{r}')\frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|-i\omega t}}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

$$\vec{A}(\vec{r},t)=\frac{\mu}{4\pi}\int_{V'}\vec{J}_f(\vec{r}')\frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|-i\omega t}}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

$$=\frac{\mu}{4\pi}\int_{V'}\vec{J}_f(\vec{r}')\frac{e^{-i\omega\left(t-\frac{|\vec{r}-\vec{r}'|}{c}\right)}}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

$$\boxed{t'=t-\frac{|\vec{r}-\vec{r}'|}{c}}$$

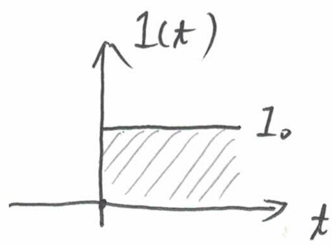
↓ 从 $\vec{r}'$ 点跑到 $\vec{r}$ 点所需要的时间

源在 t' 处辐射场，到达 $\vec{r}$ 点需要一定的时间.

⟹ 称上述解为推迟势解.

可严格求解模型

一根导线: $I(t)=\begin{cases}0, & t\le 0\\ I_0, & t>0\end{cases}$



求电场激发的电磁场分布.

t 非常大时，$\vec{B}$, $\vec{E}$ 等应该回到静态情况

$t\to-\infty$ 时，$\vec{B}$, $\vec{E}$ 等为 0

$t\geq 0$ 附近，$\vec{B}$, $\vec{E}$ 从某种方式被产生了

算电场，既需要知道 φ，又需要知道 $\vec{A}$

算磁场，只需要知道 $\vec{A}$

Steady state. $\rho_f = 0$ 时，$\varphi = 0$，磁场，电场均简单.

$$I(t) = \sum_{\omega} e^{-i\omega t} I(\omega) = \int \frac{d\omega}{2\pi} e^{-i\omega t} I(\omega)$$

为了使用推迟势公式

$$\Longrightarrow I(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} I(t)$$

$$= I_0 \int_0^{+\infty} dt\, e^{i\omega t} = I_0 \frac{e^{i\omega t}}{i\omega}\Bigg|_0^{+\infty}$$

$$= I_0 \frac{e^{i\omega t - 0_+ t}}{i(\omega + i0_+)}\Bigg|_0^{+\infty}$$

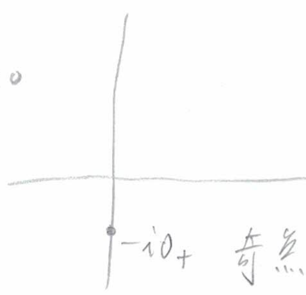
$$= i I_0 \frac{1}{\omega + i0_+}$$

$$-i(z_1 + i z_2)t = -iz_1 + z_2 t$$

验证：$I(t) = iI_0 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{1}{\omega + i0_+}$ （ω：复数） $\Longrightarrow$ **留数定理** $\Rightarrow$ 选围道.

$t<0$ 时，必须选上半平面围道，$I(t) = 0$

$t>0$ 时，必须选下半平面围道



$$I(t) = 2\pi i \frac{(iI_0)}{2\pi} e^{-i(-i0_+)t}$$

$$= I_0 e^{-0_+ t} = I_0$$

$\Longrightarrow \quad I(\omega) = -i I_0 \dfrac{1}{\omega + i0_+}\,\delta(x)\,\delta(y)$

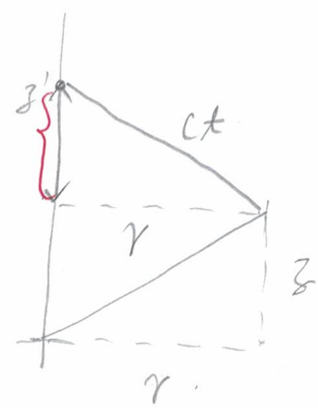
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\sum_{\omega}\int dz'\,dx'\,dy'\,(\bullet i I_0)\,\frac{1}{\omega+i0_+}\,\delta(x')\,\delta(y') \times \frac{e^{-i\omega\left(t-\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}/c\right)}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= \frac{\mu_0}{4\pi}\int\frac{d\omega}{2\pi}\int_{-L/2}^{L/2} dz'\,(\bullet i I_0)\,\frac{1}{\omega+i0_+}\,\frac{e^{-i\omega\left(t-\sqrt{x^2+y^2+(z-z')^2}/c\right)}}{\sqrt{x^2+y^2+(z-z')^2}}$$

$$\overset{x^2+y^2\equiv\rho^2}{=} \frac{\mu_0}{4\pi}\int\frac{d\omega}{2\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} dz'\,(\bullet i I_0)\,\frac{1}{\omega+i0_+}\,\frac{e^{-i\omega\left(t-\sqrt{\rho^2+(z-z')^2}/c\right)}}{\sqrt{\rho^2+(z-z')^2}}$$

1°, 当 $ct < \sqrt{\rho^2+(z-z')^2} \Longrightarrow (z-z')^2 > (ct)^2-\rho^2$

$\Longrightarrow |z-z'| > \sqrt{(ct)^2-\rho^2}$



$\Longrightarrow$ ① 当 $z'>z$ 时, $z-z' < -\sqrt{(ct)^2-\rho^2}$

即 $z' > z + \sqrt{(ct)^2-\rho^2}$

② 当 $z'<z$ 时, $z-z' > \sqrt{(ct)^2-\rho^2}$

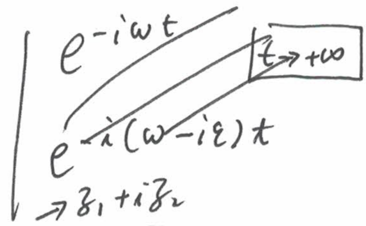
即 $z' < z - \sqrt{(ct)^2-\rho^2}$

2°, 当 $ct > \sqrt{\rho^2+(z-z')^2} \Longrightarrow |z-z'| < (ct)^2-\rho^2$

$\Longrightarrow$ ① 当 $z'>z$ 时, $z'-z<\sqrt{(ct)^2-\rho^2} \Longrightarrow z<z'<\sqrt{(ct)^2-\rho^2}+z$

② 当 $z'<z$ 时, $z-z'<\sqrt{(ct)^2-\rho^2} \Longrightarrow z-\sqrt{(ct)^2-\rho^2}<z'<z$

选择下围道条件



当 $ct>\sqrt{\rho^2+(z-z')^2}$ 时

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{d\omega}{2\pi}\int_z^{\sqrt{c^2t^2-\rho^2}+z}dz'\,(0,i I_0)\,\frac{1}{\omega+i0_+}\,\frac{e^{-i\omega\left(t-\sqrt{\rho^2+(z-z')^2}/c\right)}}{\sqrt{\rho^2+(z-z')^2}}$$

$$+\frac{\mu_0}{4\pi}\int\frac{d\omega}{2\pi}\int_{z-\sqrt{c^2t^2-\rho^2}}^{z}dz'\,(0,i I_0)\,\frac{1}{\omega+i0_+}\,\frac{e^{-i\omega\left(t-\sqrt{\rho^2+(z-z')^2}/c\right)}}{\sqrt{\rho^2+(z-z')^2}}$$

$$=\frac{\mu_0 I_0}{4\pi}\int_0^{\sqrt{c^2t^2-\rho^2}}\frac{1}{\sqrt{\rho^2+z'^2}}\,dz'$$

$$+\frac{\mu_0 I_0}{4\pi}\int_{-\sqrt{c^2t^2-\rho^2}}^{0}\frac{1}{\sqrt{\rho^2+z'^2}}\,dz'$$

$$=\frac{\mu_0 I_0}{2\pi}\int_0^{\sqrt{c^2t^2-\rho^2}}\frac{1}{\sqrt{\rho^2+z'^2}}\,dz'$$

$$=\frac{\mu_0 I_0\vec{e}_z}{2\pi}\ln\left(\sqrt{\rho^2+z'^2}+z'\right)\Bigg|_0^{\sqrt{c^2t^2-\rho^2}}$$

$$=\frac{\mu_0 I_0}{2\pi}\vec{e}_z\ln\left(\frac{ct+\sqrt{c^2t^2-\rho^2}}{\rho}\right)$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi\sqrt{c^2t^2-\rho^2}} \vec{e_z}$$

$$\vec{B} = \nabla\times\vec{A} = -\frac{\partial A_z}{\partial r}\vec{e_\varphi} = \frac{\mu_0 I_0}{2\pi r}\frac{ct}{\sqrt{c^2t^2-\rho^2}}\vec{e_\varphi}$$

分析对t的依赖

固定一点 $\rho = \rho_0$

⇑ 成立条件 $ct > \sqrt{\rho^2+(z-z')^2}$

当 $ct < \sqrt{\rho^2+(z-z')^2}$ 时 ⟹ 作业！

多极矩展开

是否需要分别求出 $\vec{A}$ 和 φ，再求 $\vec{B}$ 和 $\vec{E}$？

对 harmonic 场，可利用 Lorentz 条件

$$\nabla\cdot\vec{A} + \mu\varepsilon\frac{\partial\varphi}{\partial t} = 0$$

$$\Longrightarrow \nabla\cdot\vec{A} - i\mu\varepsilon\omega\varphi = 0 \Longrightarrow \boxed{\varphi = \frac{1}{i\mu\varepsilon\omega}\nabla\cdot\vec{A}}$$

实际电场可由 Maxwell 方程

$$\nabla\times\vec{B} = \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} = -i\frac{\omega}{c^2}\vec{E} \quad \text{定出}$$

$$\Longrightarrow \boxed{\vec{E} = \frac{ic^2}{\omega}\nabla\times\vec{B}}$$