电动力学讲稿（二）

2022.4.15

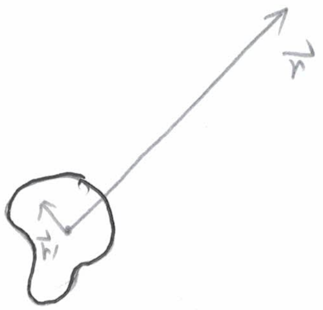
推迟势的多极矩展开

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} \vec{J}_f(\vec{r}\,')\frac{\boxed{e^{ik|\vec{r}-\vec{r}\,'|}}}{\underline{|\vec{r}-\vec{r}\,'|}}\,d\vec{r}\,'$$

$\Rightarrow$ 指数因子怎么办

$\Rightarrow$ 已处理过

Current region is small in comparison to the wavelength.



$$\lambda = \frac{2\pi c}{\omega}$$

d: current region.

① Near (static) zone: $d \ll r \ll \lambda$

② intermediate (induction) zone: $d \ll r \sim \lambda$

③. far (radiation) zone: $d \ll \lambda \ll r$

$\Rightarrow$ 这些条件应该自然从推导中得出.

① Near zone.

$$kr = \frac{2\pi}{\lambda}r \ll 1,\quad e^{ik|\vec{r}-\vec{r}\,'|} \to 1$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} J_f(\vec{r}\,')\frac{1}{|\vec{r}-\vec{r}\,'|}\,d\vec{r}\,'$$

由于 $r \gg d$，此时仍然可以用静场时学过的多极矩展开来计算.

③. In the far zone: $kr \gg 1$, the exponential oscillates rapidly, which determines the behavior of the vector potential

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$$|\vec{r} - \vec{r}'| = r - \vec{n} \cdot \vec{r}'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \left(\frac{e^{ikr}}{r} \right) \int_{V'} \vec{J}(\vec{r}')\, e^{-ik\vec{n}\cdot\vec{r}'}\, d\vec{r}'$$

⇓ 球面波

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int_{V'} \vec{J}(\vec{r}') \sum_{n=0}^{\infty} \frac{1}{n!} (\underbrace{-ik\vec{n}\cdot\vec{r}'}_{ⓚ})^n$$

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int_{V'} \vec{J}(\vec{r}') \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} (\underbrace{k\vec{n}\cdot\vec{r}'}_{kd \ll 1})^n\, d\vec{r}'$$

Intermediate zone 的级数解较为复杂，可参见 Jackson 书

下面我们重点分析 far zone，重点考察级数解前两项

$n=0$ 时

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \underbrace{\int_{V'} \vec{J}(\vec{r}')\, d\vec{r}'}_{}$$

$$\int_V \vec{J}(\vec{r}')\, d\vec{r}' = ?$$

$$\int \nabla' \cdot (x_i' \cdot \vec{J}(\vec{r}'))\, d\vec{r}' = 0 \Rightarrow \nabla\cdot\vec{J} = 0 \text{ 不再满足？}$$

$$= \int \nabla' x_i' \cdot \vec{J}(\vec{r}')\, d\vec{r}' + \int x_i' \nabla \cdot \vec{J}(\vec{r}')\, d\vec{r}'$$

$$\int J_i(\vec{r}')\,d\vec{r}' = -\int x_i'\,\nabla\cdot\vec{J}(\vec{r}')\,d\vec{r}'$$

$$\Rightarrow \int \vec{J}(\vec{r}')\,d\vec{r}' = -\int \vec{r}'\,\nabla\cdot\vec{J}(\vec{r}')\,d\vec{r}'$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{J} = 0$$

$$\Rightarrow \nabla\cdot\vec{J} = i\omega\rho$$

$$\Rightarrow \int \vec{J}(\vec{r}')\,d\vec{r}' = -i\omega\underbrace{\int \vec{r}'\rho(\vec{r}')\,d\vec{r}'}_{'''\vec{p}\text{ 电偶极矩}} = -i\omega\vec{p} = \dot{\vec{p}}$$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\,\dot{\vec{p}}$$

$$\vec{B} = \nabla\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\,\vec{e}_\phi \\ \frac{\partial}{\partial r} & 0 & 0 \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

怎么算?

$\vec{r}$ 的方向写为 $\vec{n}$

$$= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}(rA_\theta)\,r\sin\theta\,\vec{e}_\phi - \frac{\partial}{\partial r}(r\sin\theta A_\phi)\,r\vec{e}_\theta$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)\,\vec{e}_\phi - \frac{\partial}{\partial r}(rA_\phi)\,\vec{e}_\theta\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(A_\theta\vec{e}_\phi - A_\phi\vec{e}_\theta\right)\right]$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r}\frac{\partial}{\partial r}\left(e^{ikr}\right)\left(\dot{p}_\theta\vec{e}_\phi - \dot{p}_\phi\vec{e}_\theta\right)$$

$$_r \times \dot{\vec{p}} = \underline{\vec{e}_r \times \left(\dot{p}_r \vec{e}_r + \dot{p}_\theta \vec{e}_\theta + \dot{p}_\phi \vec{e}_\phi \right)}$$

$$\{r, \theta, \phi\} \Longrightarrow \vec{e}_r \times \vec{e}_\theta = \vec{e}_\phi$$

$$\vec{e}_r \times \vec{e}_\phi = -\vec{e}_\theta$$

$$= \dot{p}_\theta \vec{e}_\phi - \dot{p}_\phi \vec{e}_\theta$$

$$\Longrightarrow \vec{B} = \nabla \times \vec{A} = \frac{\mu_0}{4\pi} \frac{1}{r} \frac{\partial}{\partial r} \left(e^{ikr} \right) \vec{e}_r \times \dot{\vec{p}}$$

$$= \frac{\mu_0}{4\pi} \frac{ik}{r} e^{ikr} \vec{e}_r \times \dot{\vec{p}}$$

$$= -\frac{\mu_0}{4\pi} \frac{ik}{r} e^{ikr} \frac{1}{i\omega} \vec{e}_r \times \ddot{\vec{p}}$$

$$= \frac{\mu_0}{4\pi} \frac{k}{\omega} \frac{1}{r} e^{ikr} \ddot{\vec{p}} \times \vec{e}_r$$

$$= \frac{1}{4\pi \varepsilon_0 c^3 r} e^{ikr} \ddot{\vec{p}} \times \vec{e}_r$$

$$\boxed{\begin{array}{l} -i\omega p = \dot{\vec{p}} \\ -i\omega \dot{\vec{p}} = \ddot{\vec{p}} \end{array}}$$

$$\omega = ck$$

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

$$\vec{E} = \frac{ic}{k} \underline{\nabla \times \vec{B}}$$

为了更好计算，我们总结一下技巧.

$$\vec{B} = \nabla \times \vec{f} = \frac{1}{r} \frac{\partial}{\partial r} \left[r g(r) \left(d_\theta \vec{e}_\phi - d_\phi \vec{e}_\theta \right) \right]$$

$$\vec{f} = g(r) \cdot \vec{d} \quad \Bigg| = \underbrace{\frac{1}{r} \frac{\partial}{\partial r} \left(r g(r) \right) \vec{e}_r \times \vec{d}}_{\text{这样一个简单公式}}$$

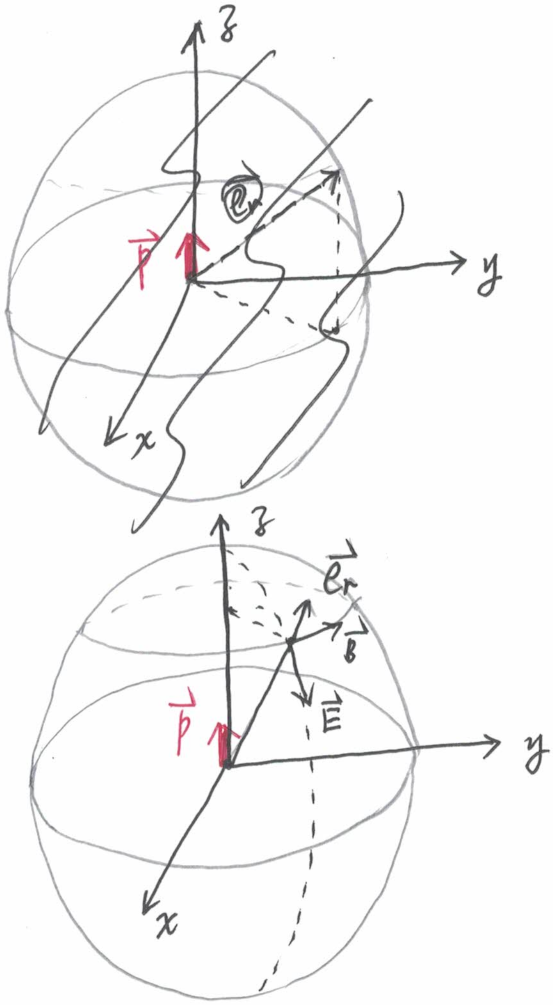
$$\vec{E} = \frac{ic}{k} \nabla \times \vec{B}$$

$$= \frac{ic}{k} \frac{1}{4\pi\varepsilon_0 c^3} \frac{\partial}{\partial r}\left(e^{ikr}\right) \vec{e}_r \times \left(\ddot{\vec{p}} \times \vec{e}_r\right)$$

$$= -\frac{1}{4\pi\varepsilon_0 c^2} e^{ikr} \vec{e}_r \times \left(\ddot{\vec{p}} \times \vec{e}_r\right)$$

$$= \boxed{\frac{1}{4\pi\varepsilon_0 c^2} e^{ikr} \left(\ddot{\vec{p}} \times \vec{e}_r\right) \times \vec{e}_r}$$

若取球坐标原点在电荷分布区内，$\vec{p}$方向为极轴



$\vec{B} \sim \vec{p} \times \vec{e}_r$

$\vec{E} \sim \vec{B} \times \vec{e}_r$

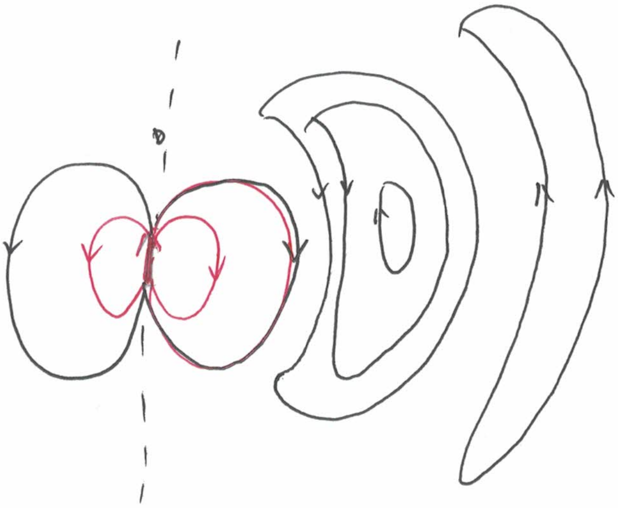
$\vec{B}$ 沿纬线振荡

$\vec{E}$ 沿经线振荡

$$\vec{B} = \frac{1}{4\pi\varepsilon_0 c^3 r} \ddot{p} e^{ikr} \sin\theta \, \vec{e}_\phi$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 c^2 r} \ddot{p} e^{ikr} \sin\theta \, \vec{e}_\theta$$

$\vec{E}$ 的分布



一些特点

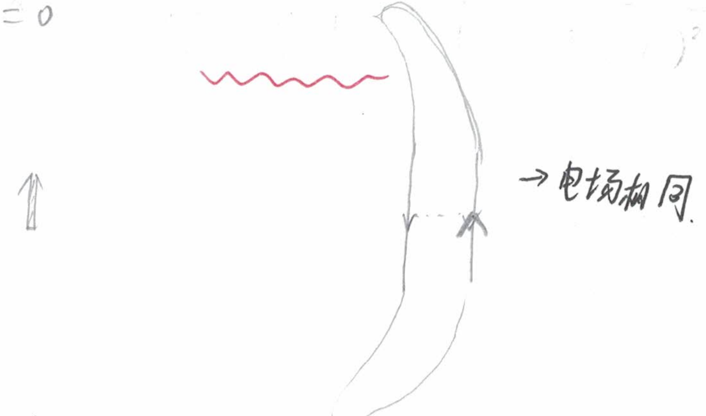
① $\frac{|E|}{|B|} = c \Longrightarrow$ 电场强度与磁场强度的比值为c

电场与磁场始终垂直

② $\vec{B}, \vec{E} \propto \sin\theta \Longrightarrow$ 与电偶极子垂直方向辐射最强，平行方向辐射为零

③. 电场线闭合，在这一阶时横向（考虑高阶会有修正）

$\hookrightarrow \nabla\cdot\vec{E} = 0$



可能可以用map图表示出来

辐射能流，辐射功率

$\vec{B} \propto e^{ikr-i\omega t}$ ，取实部

$\vec{E} \propto e^{ikr-i\omega t}$ ，取实部

$$\vec{B}_{实} = \frac{1}{2}\left(\vec{B} + \vec{B}^*\right)$$

$$\vec{E}_{实} = \frac{1}{2}\left(\vec{E} + \vec{E}^*\right)$$

能流密度：

$$\vec{S} = \vec{E}_{实} \times \vec{H}_{实}$$

$$= \frac{1}{4}\left(\vec{E} + \vec{E}^*\right) \times \left(\vec{H} + \vec{H}^*\right)$$

$$= \frac{1}{4}\Big(\underbrace{\vec{E}\times\vec{H} + \vec{E}^*\times\vec{H}^*}_{e^{-2i\omega t}} + \underbrace{\overset{A}{\vec{E}\times\vec{H}^*} + \overset{A^*}{\vec{E}^*\times\vec{H}}}_{与时间无关}\Big)$$

$$\boxed{\vec{S}_{平均} = \frac{1}{2}\,\mathrm{Re}\left(\vec{E}^*\times\vec{H}\right)}$$

$\Downarrow$ 即平均能流公式，与时间无关

$\vec{p} = p_a e^{-i\omega t}$

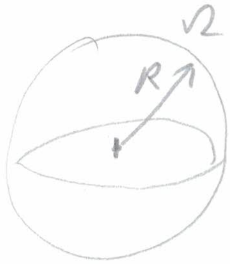
$$= \frac{1}{2\mu_0}\frac{\omega^2}{(4\pi\varepsilon_0 c^2 r)^2 c}\, p_a^2 \sin^2\theta \; \vec{e}_\theta \times \vec{e}_\phi$$

辐射的角分布

$\Uparrow$

$$= \frac{\omega^2}{32\pi^2\varepsilon_0 c^3 r^2}\, p_a^2 \sin^2\theta \; \vec{e}_r = \frac{1}{32\pi^2\varepsilon_0 c^3 r^2}\left|\ddot{\vec{p}}\right|^2 \boxed{\sin^2\theta}\, \vec{e}_r$$

辐射总功率：单位时间通过某一球面的能量



$$P = \oint_{\Omega} d\vec{S}\,|\vec{S}|$$

$$= \int R^2 d\Omega \,\frac{1}{32\pi^2\varepsilon_0 c^3 R^2}|\ddot{p}|^2\sin^2\theta$$

$$= \int \sin\theta\, d\theta\, d\phi \,\sin^2\theta$$

$$= \int d\Omega\,\frac{1}{32\pi^2\varepsilon_0 c^3}|\ddot{p}|^2\sin^2\theta = \frac{|\ddot{p}|^2}{32\pi^2\varepsilon_0 c^3}\underbrace{\int d\Omega \sin^2\theta}$$

$$-\int_0^{2\pi} d\phi \int_0^{2\pi} d\cos\theta\,(1-\cos^2\theta) = -2\pi\left(\cos\theta - \frac{\cos^3\theta}{3}\right)\Big|_0^{\pi}$$

$$= -2\pi\left(-1+\frac{1}{3}\right)(-1)\times 2 = 2\pi\times\frac{4}{3} = \frac{8\pi}{3}$$

$$= \frac{|\ddot{p}|^2}{12\pi\varepsilon_0 c^3}$$ ✓ ☜

⟹ 辐射功率正比于频率的四次方，ω 增大，辐射迅速增强

⟹ 启示：材料本身可以产生 $\vec{p}$ ⟹ $\vec{p}$ 的辐射过程可以由上面分析给出？

$$\vec{P} = \vec{p}_0 + \vec{p}(t)$$

作业：运用 Maxwell 方程组求解介电体 $\vec{p}(t)$ 的辐射公式，并研究其辐射能流及辐射总功率.

$$\boxed{\text{接②页中，}n=1\text{ 时}}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int_{V'}\vec{J}(\vec{r}')(-i)(k\vec{n}\cdot\vec{r}')\,d\vec{r}'$$

$$= \frac{-ik\mu_0}{4\pi r}e^{ikr}\int_{V'}\vec{J}(\vec{r}')\underbrace{(\vec{n}\cdot\vec{r}')}_{\text{“电流”的矩.}}\,d\vec{r}'$$

下面我们将看到，上式代表磁偶极矩及电四极矩产生的辐射.

$$\Downarrow \qquad\qquad \Downarrow$$

$$\vec{m} \qquad\qquad Q_{ij}$$

$$i\omega\rho = \nabla\cdot\vec{J}$$

$$\boxed{(\vec{n}\cdot\vec{r}')\,\vec{J} = ?}$$

$$= (\vec{n}\cdot\vec{r}')\vec{J}$$

$$= \frac{1}{2}\,\vec{n}\cdot\vec{r}'\otimes\vec{J} + \frac{1}{2}\,\vec{n}\cdot\vec{J}\otimes\vec{r}'$$

$$+\frac{1}{2}\,\vec{n}\cdot\vec{r}'\otimes\vec{J} - \frac{1}{2}\,\vec{n}\cdot\vec{J}\otimes\vec{r}'$$

✓ $$\int_{V'}\vec{J}(\vec{r}')(\vec{n}\cdot\vec{r}')\,d\vec{r}'$$

$$= \frac{1}{2}\int\left[(\vec{n}\cdot\vec{r}')\vec{J} + (\vec{n}\cdot\vec{J})\vec{r}'\right]d\vec{r}' \qquad \underline{①}$$

$$+\frac{1}{2}\int\left[(\vec{n}\cdot\vec{r}')\vec{J} - (\vec{n}\cdot\vec{J})\vec{r}'\right]d\vec{r}' \qquad \underline{②}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

②项

$$(\vec{n}\cdot\vec{r}\,')\vec{J}-(\vec{n}\cdot\vec{J})\vec{r}\,'$$

$$=\vec{n}\times(\vec{J}\times\vec{r}\,')=-\vec{n}\times(\vec{r}\,'\times\vec{J})$$

$$\vec{A}_{磁}=\frac{-ik\mu_0}{4\pi r}e^{ikr}(-\vec{n})\times\underbrace{\frac{1}{2}\int_{V'}\vec{r}\,'\times\vec{J}(\vec{r}\,')\,d\vec{r}\,'}_{\vec{m}}$$

$$=\frac{ik\mu_0}{4\pi r}e^{ikr}\,\vec{n}\times\vec{m}$$

$\Longrightarrow$ 磁偶极辐射.

①项:

$$\frac{1}{2}\int d\vec{r}\,'\left[(\vec{n}\cdot\vec{r}\,')\vec{J}+(\vec{n}\cdot\vec{J})\vec{r}\,'\right]\Rrightarrow\frac{\vec{r}}{r}\cdot\int d\vec{r}\,'\,\vec{r}\,'\vec{J}(\vec{r}\,')$$

~~前面(一) 64 页已经证明~~

~~$\vec{r}\cdot\int\vec{r}\,'\vec{J}_i=-\frac{1}{2}\left[\vec{r}\times\int(\vec{r}\,'\times\vec{J})d\vec{r}\,'\right]_i$~~

$$=\boxed{\frac{1}{2r}\int d\vec{r}\,'\left(x_j x_j'\vec{J}+x_j J_j\vec{r}\,'\right)}$$

i分量

$$\frac{1}{2r}\int d\vec{r}\,'\left(x_j x_j' J_i+x_j J_j x_i'\right)$$

$$=\frac{x_j}{2r}\underbrace{\int d\vec{r}\,'\left(x_j' J_i+x_i' J_j\right)}$$

↗ 因 $\nabla\cdot\vec{J}=0$

前面在静磁场 63 页做过，那时只要

还是重复以前的过程

$$\int \nabla'\cdot(fg\vec{J})\,d\vec{r}\,' = 0 \,,\quad \text{if } \vec{J} \text{ is localized}$$

$$\Rightarrow \int [g\nabla' f\cdot\vec{J} + f\nabla' g\cdot\vec{J} + fg\nabla'\cdot\vec{J}]\,d\vec{r}\,'$$

令 $f = x_i'$, $g = x_j'$, $\nabla'\cdot\vec{J} = \cancel{0}\,\underline{i\omega\rho}\ \boxed{= -\frac{\partial\rho}{\partial t}}$

$$= \int [x_j' J_i + x_i' J_j + x_i' x_j' i\omega\rho]\,d\vec{r}\,' = 0$$

$$\Rightarrow \int [x_j' J_i + x_i' J_j]\,d\vec{r}\,' = -\int x_i' x_j' i\omega\rho\, d\vec{r}\,'$$

$\Rightarrow$ i分量: $\frac{1}{2}\int d\vec{r}\,' [(\vec{n}\cdot\vec{r}\,')\vec{J} + (\vec{n}\cdot\vec{J})\vec{r}\,']_i$

$$= -\frac{x_j}{2r}\int x_i' x_j' i\omega\rho(\vec{r}\,')\,d\vec{r}\,'$$

$$= -\frac{1}{2r}\int x_i'(\vec{r}\cdot\vec{r}\,')\,i\omega\rho(\vec{r}\,')\,d\vec{r}\,'$$

$\Rightarrow$ 整体: $\frac{1}{2}\int d\vec{r}\,' [(\vec{n}\cdot\vec{r}\,')\vec{J} + (\vec{n}\cdot\vec{J})\vec{r}\,']$

$$= -\frac{i\omega}{2}\int \vec{r}\,'(\vec{n}\cdot\vec{r}\,')\rho(\vec{r}\,')\,d\vec{r}\,' \quad ✓$$

$$\vec{A}_{电} = \frac{-ik\mu_0}{4\pi r}e^{ikr}\left(-\frac{i\omega}{2}\right)\int \vec{r}\,'(\vec{n}\cdot\vec{r}\,')\rho(\vec{r}\,')\,d\vec{r}\,'$$

$$= -\frac{k\mu_0\omega}{8\pi r}e^{ikr}\int \underset{\smile}{\vec r'}(\vec n\cdot\underset{\smile}{\vec r'})\underset{\sim}{\rho(\vec r')}\,d\vec r'$$

ρ 的二阶矩，可用电四级矩表出！

$$= -\frac{k\mu_0\omega}{8\pi r}e^{ikr}\,\underset{\sim\sim\sim}{\vec n\cdot\int}\vec r'\vec r'\rho(\vec r')\,d\vec r'$$

回顾电四极矩定义

$$Q_{ij}=\int\left(3x_i'x_j'-r'^2\delta_{ij}\right)\rho(\vec r')\,d\vec r'$$

$$= -\frac{k\mu_0\omega}{8\pi r}e^{ikr}\sum_j\frac{x_j}{r}\int x_j'\vec r'\rho(\vec r')\,d\vec r'$$

$$= -\frac{k\mu_0\omega}{8\pi r}e^{ikr}\sum_{ij}\frac{x_j}{r}\int x_j'x_i'\rho(\vec r')\,d\vec r'\,\vec e_i$$

$$= -\frac{k\mu_0\omega}{8\pi r}e^{ikr}\sum_{ij}\frac{x_j}{r}\frac{1}{3}\int\left(3x_j'x_i'-r'^2\delta_{ij}\right)\rho(\vec r')\,d\vec r'\,\vec e_i$$

$$\underbrace{-\frac{k\mu_0\omega}{8\pi r}e^{ikr}\sum_{ij}\frac{x_j}{r}\frac{1}{3}\int r'^2\delta_{ij}\rho(\vec r')\,d\vec r'\,\vec e_i}$$

$$= -\frac{k\mu_0\omega}{8\pi r}e^{ikr}\sum_i\frac{x_i}{r}\frac{1}{3}\int r'^2\rho(\vec r')\,d\vec r'\,\vec e_i \Rightarrow$$

后面这项沿 $\vec r$ 方向.

$$\Longrightarrow \vec{A}_{电} = -\frac{k\mu_0\omega}{24\pi r} e^{ikr}\sum_{ij}\frac{x_j}{r}Q_{ij}\vec{e_i}$$

$$-\frac{k\mu_0\omega}{24\pi r}e^{ikr}\vec{n}\int r'^2\rho(\vec{r}')d\vec{r}'$$

也是非常
简洁的表达式

$\Longrightarrow$ 总的矢势：

$$\vec{A} = \vec{A}_{磁} + \vec{A}_{电}$$

$$= \frac{ik\mu_0}{4\pi r}e^{ikr}\left(\vec{n}\times\vec{m} + \frac{i\omega}{6}\sum_{ij}\overset{=n_j}{\boxed{\frac{x_j}{r}}}Q_{ij}\vec{e_i} + \underbrace{\frac{i\omega}{6}\vec{n}\int r'^2\rho(\vec{r}')d\vec{r}'}\right)$$

这一项对磁场无贡献

$$\vec{B} = \nabla\times\vec{A}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{ik\mu_0}{4\pi}e^{ikr}\right)\vec{n}\times(\vec{n}\times\vec{m})$$

$$+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{ik\mu_0}{4\pi}e^{ikr}\right)\vec{n}\times\left(\frac{i\omega}{6}\sum_{ij}n_j Q_{ij}\vec{e_i}\right)$$

$$= -\frac{1}{r}\frac{k^2\mu_0}{4\pi}e^{ikr}\vec{n}\times(\vec{n}\times\vec{m})$$

$$-\frac{1}{r}\frac{k^2\mu_0}{4\pi}e^{ikr}\frac{i\omega}{6}\vec{n}\times\left(\sum_{ij}n_j Q_{ij}\vec{e_i}\right)$$

$$\vec{E} = \frac{ic}{k}\nabla\times\vec{B}$$

下面分【磁偶极辐射】及【电四极辐射】分析辐射性质

1). 磁偶极辐射

$$\vec{B}_{磁} = -\frac{1}{r}\frac{c^2k^2\mu_0}{4\pi c^2} e^{ikr}\underbrace{\vec{n}\times(\vec{n}\times\vec{m})}_{(\vec{m}\cdot\vec{n})\vec{n}-\vec{m}}$$

$$= \frac{\mu_0}{4\pi c^2 r} e^{ikr}\,\vec{n}\times(\vec{n}\times\ddot{\vec{m}})$$

$$= \frac{\mu_0}{4\pi c^2 r} e^{ikr}\left((\ddot{\vec{m}}\cdot\vec{n})\vec{n}-\ddot{\vec{m}}\right)$$

$$\vec{E}_{磁} = \frac{ic}{k}\nabla\times\vec{B}_{磁}$$

$$= \frac{\mu_0}{4\pi c^2 r}\frac{ic}{k}\,ik\,e^{ikr}(-\vec{n}\times\ddot{\vec{m}})$$

$$= \frac{\mu_0}{4\pi c r} e^{ikr}\,\vec{n}\times\ddot{\vec{m}}$$

平均辐射能流：

$$\vec{S}_{平均} = \frac{1}{2}\mathrm{Re}\,(\vec{E}^*_{磁}\times\vec{H}_{磁}) \qquad \boxed{\vec{m} = \vec{m}_\alpha e^{-i\omega t}}$$

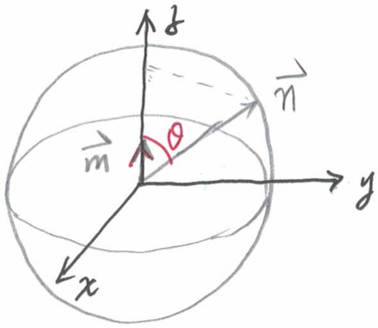
$$\frac{1}{2}\vec{E}^*_{磁}\times\vec{H}_{磁}$$

$$= \frac{1}{2}\frac{\mu_0(-\omega^2)}{4\pi c r}(\vec{n}\times\vec{m}_\alpha)\times\frac{(-\omega^2)}{4\pi c^2 r}\left[(\vec{m}_\alpha\cdot\vec{n})\vec{n}-\vec{m}_\alpha\right]$$

$$= \frac{1}{2}\frac{\mu_0\omega^4}{16\pi^2c^3r^2}(\vec{n}\times\vec{m}_\alpha)\times\left[(\vec{m}_\alpha\cdot\vec{n})\vec{n}-\vec{m}_\alpha\right]$$

$\boxed{\vec{m},\ 沿z轴}$ ⟹ 电动力学需要培养学生做计算能力

$$\Longrightarrow \vec{S}_{平均} = \frac{\mu_0 \omega^4}{32\pi^2 c^3 r^2} (\vec{n} \times m_\alpha \vec{z}) \times [m_\alpha (\vec{z}\cdot\vec{n})\vec{n} - m_\alpha \vec{z}]$$



$$\boxed{\vec{n} = \cos\theta\,\vec{z} + \sin\theta\,\vec{z}_\perp}$$

$$= \frac{\mu_0 \omega^4 m_\alpha^2}{32\pi^2 c^3 r^2} \underbrace{(\vec{n} \times \vec{z}) \times [(\vec{z}\cdot\vec{n})\vec{n} - \vec{z}]}$$

$$-(\vec{z}\cdot\vec{n})\vec{n}\times(\vec{n}\times\vec{z}) + \vec{z}\times(\vec{n}\times\vec{z})$$

$$= -(\vec{z}\cdot\vec{n})[(\vec{n}\cdot\vec{z})\vec{n} - \vec{z}] + \vec{n} - (\vec{z}\cdot\vec{n})\vec{z}$$

$$= -(\vec{n}\cdot\vec{z})^2\vec{n} + \vec{n} = (1-\cos^2\theta)\vec{n}$$

$$= \sin^2\theta\,\vec{n}$$

$$= \frac{\mu_0 \omega^4 m_\alpha^2}{32\pi^2 c^3 r^2} \sin^2\theta\,\vec{n} = \frac{\mu_0 \omega^4 |\vec{m}|^2}{32\pi^2 c^3 r^2} \sin^2\theta\,\vec{n}$$

辐射功率:

$$P = \int R^2 d\Omega \frac{\mu_0 \omega^4 |\vec{m}|^2}{32\pi^2 c^3 R^2} \sin^2\theta = \frac{\mu_0 \omega^4 |\vec{m}|^2}{12\pi c^3}$$

计算（略）

作业：一电流线圈半径为a，电流 $I_0 e^{-i\omega t}$，求辐射功率.

2). 电四极辐射

$$\vec{B}_{电} = -\frac{1}{r}\frac{k^2\mu_0}{4\pi}e^{ikr}\frac{i\omega}{6}\vec{n}\times\left(\sum_{ij} n_j Q_{ij}\vec{e}_i\right)$$

记 $\mathcal{D}_i = \sum_j n_j Q_{ij}$ $\boxed{这是一个矢量}$

$$\vec{\mathcal{D}} = \sum_i \mathcal{D}_i \vec{e}_i$$ ⟹ 这样书写理论就会比较优美

$$= -\frac{1}{r}\frac{k^2\mu_0}{4\pi}e^{ikr}\frac{i\omega}{6}\vec{n}\times\vec{\mathcal{D}}$$

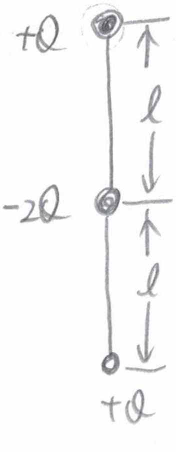
$$= -\frac{i\omega k^2\mu_0}{24\pi r}e^{ikr}\vec{n}\times\vec{\mathcal{D}}$$ ✓

$$\vec{E}_{电} = \frac{\omega k^3\mu_0}{24\pi r}e^{ikr}\vec{n}\times(\vec{n}\times\vec{\mathcal{D}})$$

$$= \frac{\omega k^3\mu_0}{24\pi r}e^{ikr}\left[(\vec{n}\cdot\vec{\mathcal{D}})\vec{n}-\vec{\mathcal{D}}\right]$$

辐射平均能流密度易算出（作业）

具体考虑一个电四极子



求出 $\vec{\mathcal{D}}$，计算 $\vec{B}_{电}$，$\vec{E}_{电}$ 及平均能流密度（作业）

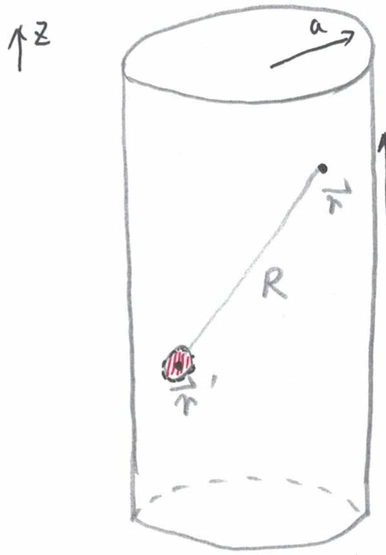
天线辐射

天线上的电流分布

$\vec{J}(\vec{r}) \Longleftarrow \vec{J}(\vec{r})$ 是如何分布的？

A complicated boundary value problem

考虑一圆柱形导体，半径为 a，长度为 d



$\boxed{I_z\overset{(\rho=a,z)}{(z)}}$

$a \ll d, \lambda$. $\Rightarrow$ "无限长"

perfect conducting $\sigma \to \infty$.

↳ 电流只在导体表面 $\boxed{\text{流动}}$

$I_z \to A_z$，我们关于 [illegible] 如何依赖于子

$$\boxed{\vec{A} = \hat{z} A_z \cancel{(\rho = a, z)}} \quad \hat{z} A_z(z)\, \delta(\rho - a)/\rho$$

由 Lorentz gauge:

$$\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0, \quad \varphi \sim e^{-i\omega t}$$

$$\sim e^{-ickt}$$

$$\Rightarrow \varphi(\vec{r}) = -i\frac{c}{k}\nabla\cdot\vec{A}$$

电场 $\vec{E}(\vec{r}) = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}$

$$= \frac{ic}{k}\nabla(\nabla\cdot\vec{A}) + i\overset{\nearrow ck}{\omega}\vec{A} = \frac{ic}{k}\left[\nabla(\nabla\cdot\vec{A}) + k^2\vec{A}\right]$$

沿z方向的电场满足

$$E_z(\vec{r}) = \frac{ic}{k}\left[\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial z}\right) + k^2 A_z\right]$$

$$= \frac{ic}{k}\left(\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z\right)$$

沿其它方向似也有电场

$$E_x = \frac{ic}{k}\frac{\partial}{\partial x}\frac{\partial A_z}{\partial z}, \quad E_y = \frac{ic}{k}\frac{\partial}{\partial y}\frac{\partial A_z}{\partial z}$$ （有什么用？）

But on the surface of the perfectly conducting antenna the tangential component of $\vec{E}$ vanishes.

$$\Longrightarrow \left(\frac{\partial^2}{\partial z^2} + k^2\right)\underline{A_z(\rho = a, z)} = 0$$

which is an exact solution

矢势本身也由电流所产生

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

$$\vec{J}(\vec{r}) = I_z(z)\delta(\rho - a)/\rho$$

$$\Longrightarrow A_z(\vec{r}) = \frac{\mu_0}{4\pi}\int I_z(z')\frac{\delta(\rho'-a)}{\rho'}\frac{e^{ik\sqrt{(\vec{\rho}-\vec{\rho}')^2+(z-z')^2}}}{\sqrt{(\vec{\rho}-\vec{\rho}')^2+(z-z')^2}}\rho' d\rho' d\varphi' dz'$$

我们需要计算 $A_z(\rho = a, z)$

$$\sqrt{(\vec{\rho}-\vec{\rho}')^2+(z-z')^2} = \sqrt{\rho^2+\rho'^2-2\vec{\rho}\cdot\vec{\rho}'+(z-z')^2}$$

$$\overset{\rho=\rho'=a}{\Rightarrow} \sqrt{2a^2-2a^2\cos\varphi'+(z-z')^2}$$

$$A_z(\rho=a,z) = \frac{\mu_0}{4\pi}\int_0^{2\pi} I_z(z')\,\frac{e^{ik\sqrt{2a^2(1-\cos\overset{\nearrow 2\beta}{\varphi'})+(z-z')^2}}}{\sqrt{2a^2(1-\cos\varphi')+(z-z')^2}}\,\overset{\frac{1}{2}d2\beta}{d\varphi'}dz'$$

$$1-\cos\varphi' \Rightarrow \cos^2\theta-\sin^2\theta=\cos2\theta \Rightarrow 2\cos^2\theta-1=\cos2\theta$$

$$\Rightarrow 1-2\sin^2\theta=\cos2\theta \Rightarrow 1-\cos2\theta=2\sin^2\theta.$$

$$\Rightarrow A_z(\rho=a,z) = \frac{\mu_0}{8\pi}\int_0^{\pi} d\beta\int_{z_0}^{z_0+d} dz'\,I_z(z')\,\frac{e^{ik\sqrt{4a^2\sin^2\beta+(z-z')^2}}}{\sqrt{4a^2\sin^2\beta+(z-z')^2}}$$

$$= \frac{\mu_0}{4\pi}\int_{z_0}^{z_0+d} dz'\,I(z')\,K(z-z')$$

则我们有 Green function

$$K(z-z') = \frac{1}{2}\int_0^{\pi} d\beta\,\frac{e^{ik\sqrt{4a^2\sin^2\beta+(z-z')^2}}}{\sqrt{4a^2\sin^2\beta+(z-z')^2}}$$

最后我们可得出解电流分布的 $\boxed{\text{微分积分方程}}$

$$\left(\frac{d^2}{dz^2}+k^2\right)\int_{z_0}^{z_0+d} I(z')\,K(z-z')\,dz' = 0$$

要真正求解，还需要知道边界条件.

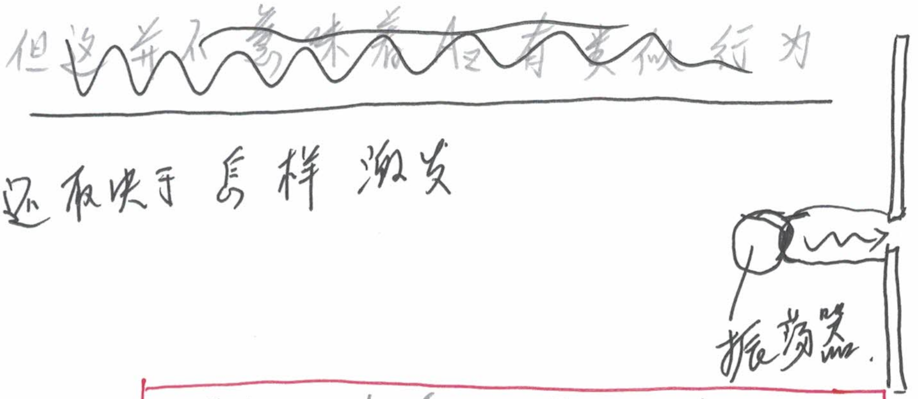
一般解：

$$a_1 \cos kz + a_2 \sin kz = \int_{z_0}^{z_0+d} I(z')\, K(z-z')\, dz'$$

天线电流在天线两端消失

$$\Longrightarrow I(z=z_0)=0, \quad I(z=z_0+d)=0$$

但这并不意味着 A_z 有类似行为



还取决于怎样激发

尝试（没有办法时取些极限情况分析）

$a \to 0$.

$$K(z-z') \to \frac{\pi}{2} \frac{e^{ik|z-z'|}}{|z-z'|}$$

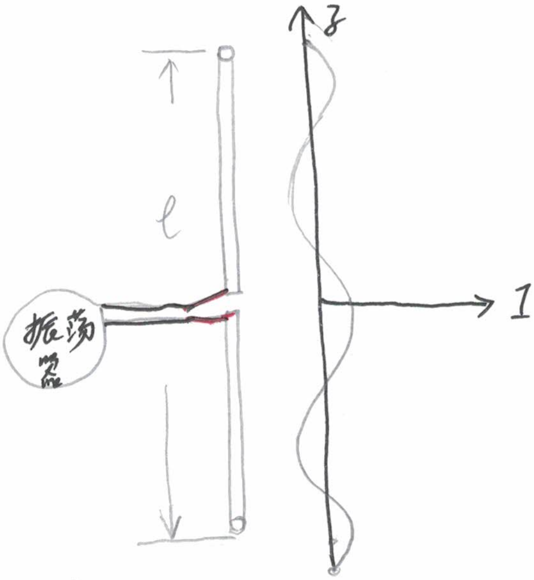
$$A_z(\rho=a, z) \to \frac{\mu_0}{8} \int_{z_0}^{z_0+d} dz'\, I(z') \frac{e^{ik|z-z'|}}{|z-z'|}$$

$$= a_1 \cos kz + a_2 \sin kz$$

~~似可用 Weyl idensity~~?

半波天线

天线上的电流近似为驻波形式，两端为波节



$$I(z) = \begin{cases} I_0 \sin k\left(\frac{l}{2} - z\right) \delta(x)\delta(y), & 0 < z \le \frac{l}{2} \\ I_0 \sin k\left(\frac{l}{2} + z\right) \delta(x)\delta(y), & -\frac{l}{2} \le z \le 0 \end{cases}$$

$$k = \frac{\omega}{c}$$

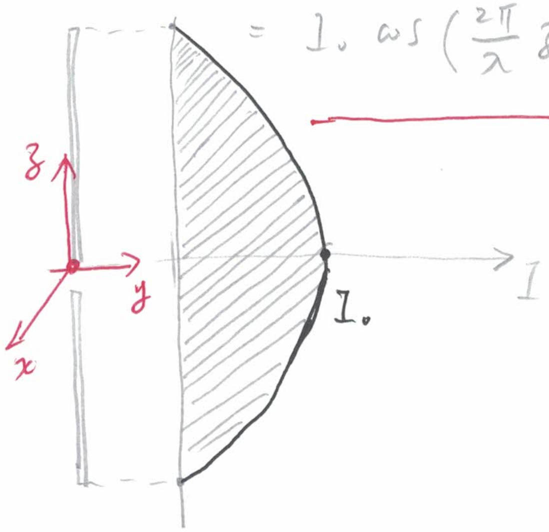
半波天线 指 $l = \frac{\lambda}{2}$，$k\frac{\lambda}{4} = \frac{1}{4} k \cdot \frac{2\pi}{k} = \frac{\pi}{2}$

$$\sin\left(k\frac{\lambda}{4} - kz\right) = \sin\left(\frac{\pi}{2} - kz\right) = \cos(kz)$$

$$\sin\left(k\frac{\lambda}{4} + kz\right) = \sin\left(\frac{\pi}{2} + kz\right) = \cos(kz)$$

$$\Longrightarrow \quad I(z) = I_0 \cos(kz)\, \delta(x)\delta(y), \quad |z| \le \frac{l}{2}$$

$$= I_0 \cos\left(\frac{2\pi}{\lambda} z\right) \delta(x)\delta(y), \quad |z| \le \frac{\lambda}{4}$$

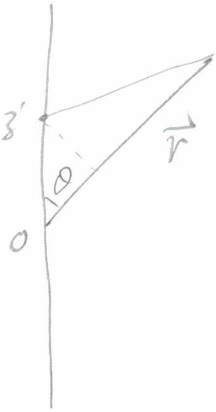


计算矢势:

$$A_z(\vec r) = \frac{\mu_0}{4\pi}\int_V \frac{\vec J(\vec r')\, e^{ik|\vec r-\vec r'|}}{|\vec r-\vec r'|}\, d\vec r'$$

$$= \frac{\mu_0}{4\pi}\int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \frac{I_0\cos\left(\frac{2\pi}{\lambda}z'\right)e^{ik\sqrt{(x^2+y^2)+(z-z')^2}}}{\sqrt{x^2+y^2+(z-z')^2}}\,dz'$$

$$\underbrace{|\vec r-\vec r'| = r-\vec n\cdot\vec r'}_{\text{计算远场}}$$



$$\overset{\text{远场}}{=} \frac{\mu_0}{4\pi}\int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \frac{I_0\cos\left(\frac{2\pi}{\lambda}z'\right)e^{ik(r-z'\cos\theta)}}{r}\,dz'$$

$\Longrightarrow$ 只保留 r

$$= \frac{\mu_0 I_0}{4\pi r}e^{ikr}\int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \cos\left(\frac{2\pi}{\lambda}z'\right)e^{-ikz'\cos\theta}\,dz'$$

$$= \frac{\mu_0 I_0}{4\pi r}e^{ikr}\int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \frac{e^{i\frac{2\pi}{\lambda}z'}+e^{-\frac{2\pi}{\lambda}z'}}{2}\,e^{-ikz'\cos\theta}\,dz'$$

$$= \frac{\mu_0 I_0}{8\pi r}e^{ikr}\int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \left[e^{i\left(\underline{\frac{2\pi}{\lambda}-k\cos\theta}\right)z'}+e^{i\left(\underline{-\frac{2\pi}{\lambda}-k\cos\theta}\right)z'}\right]dz$$

$$= \frac{\mu_0 I_0}{8\pi r}e^{ikr}\left[\left.\frac{e^{i\left(\frac{2\pi}{\lambda}-k\cos\theta\right)z'}}{i\left(\frac{2\pi}{\lambda}-k\cos\theta\right)}\right|_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}}+\left.\frac{e^{i\left(-\frac{2\pi}{\lambda}-k\cos\theta\right)z'}}{i\left(-\frac{2\pi}{\lambda}-k\cos\theta\right)}\right|_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}}\right.$$

$$= \frac{\mu_0 I_0}{8\pi r} e^{ikr} \left[\frac{e^{i\left(\frac{2\pi}{\lambda} - k\cos\theta\right)\frac{\lambda}{4}} - e^{-i\left(\frac{2\pi}{\lambda} - k\cos\theta\right)\frac{\lambda}{4}}}{i\left(\frac{2\pi}{\lambda} - k\cos\theta\right)} \right.$$

$$\left. + \frac{e^{i\left(-\frac{2\pi}{\lambda} - k\cos\theta\right)\frac{\lambda}{4}} - e^{-i\left(-\frac{2\pi}{\lambda} - k\cos\theta\right)\frac{\lambda}{4}}}{i\left(-\frac{2\pi}{\lambda} - k\cos\theta\right)} \right]$$

$$\left(\begin{array}{l} e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i \\ e^{-i\frac{\pi}{2}} = -i \end{array} \right.$$

$$= \frac{\mu_0 I_0}{8\pi r} e^{ikr} \left[\frac{i e^{-ik\cos\theta\frac{\lambda}{4}} + i e^{ik\cos\theta\frac{\lambda}{4}}}{i\left(\frac{2\pi}{\lambda} - k\cos\theta\right)} + \frac{-i e^{-ik\cos\theta\frac{\lambda}{4}} - i e^{ik\cos\theta\frac{\lambda}{4}}}{i\left(-\frac{2\pi}{\lambda} - k\cos\theta\right)} \right]$$

$$= \frac{\mu_0 I_0}{8\pi r} e^{ikr} \left[\frac{1}{\frac{2\pi}{\lambda} - k\cos\theta} + \frac{1}{\frac{2\pi}{\lambda} + k\cos\theta} \right] \left(e^{-ik\cos\theta\frac{\lambda}{4}} + e^{ik\cos\theta\frac{\lambda}{4}} \right)$$

$$= \frac{\mu_0 I_0}{8\pi r} e^{ikr} \frac{\frac{4\pi}{\lambda}}{\left(\frac{2\pi}{\lambda}\right)^2 - k^2\cos^2\theta} \, 2\cos\left(\frac{\pi}{2}\cos\theta\right)$$

$$= \frac{\mu_0 I_0}{8\pi r} e^{ikr} \frac{4k}{k^2\sin^2\theta} \cos\left(\frac{\pi}{2}\cos\theta\right)$$

$$= \boxed{\frac{\mu_0 I_0}{2\pi k r} e^{ikr} \frac{1}{\sin^2\theta} \cos\left(\frac{\pi}{2}\cos\theta\right)}$$

进一步计算出

~~电~~ 磁场：$\vec{B}(\vec{r}) = -i \frac{\mu_0 I_0 e^{ikr}}{2\pi r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \vec{e_\phi}$

电场：$\vec{E}(\vec{r}) = -i \frac{\mu_0 c I_0 e^{ikr}}{2\pi r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \vec{e_\theta}$

平均辐射能流密度

$$\overline{S} = \frac{1}{2}\mathrm{Re}\left(\vec{E}^* \times \vec{H}\right) = \frac{\mu_0 c I_0^2}{8\pi^2 r^2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \vec{e_R}$$

较集中在 $\theta = \frac{\pi}{2}$ 处

作业：1) 计算 $l = \lambda$，全波天线的辐射，并与半波天线进行对比

2). 计算半波天线阵列的远场辐射，并分析角度依赖.

电磁波的传播、衍射及散射

电磁波脱离波源之后，在空间的运动即为电磁波的传播.

简单情况：**各向同性介质中传播**

$$\begin{cases} \nabla\times\vec{H} = \vec{J}_f + \dfrac{\partial\vec{D}}{\partial t} \\ \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{H} = 0 \\ \nabla\cdot\vec{D} = \rho_f \end{cases}$$

丰富在材料上，即使是线性介质，也有各向异性

$$D_i(\omega,\vec{r}) = \int \varepsilon_{ij}(\omega,\vec{r}-\vec{r}')\,E_j(\vec{r}',$$

$$B_i(\omega,\vec{r}) = \int \mu_{ij}(\omega,\vec{r}-\vec{r}')\,H_j(\vec{r}',$$

$\boxed{\text{可}\overset{\text{且}}{\text{解}}\text{出本征振动模式}}$.

ε_{ij}，μ_{ij} 依赖于频率，也依赖于空间，

本身为张量，也是复数 $\Longrightarrow$ 导致丰富的物质响

寻找本征振动，则任何振动均为本征振动的线性叠加.

在本章，我们大多讨论各向同性介质这种简单情况，我们也忽略响应函数对空间的 retardation，即

$f(\vec{r}-\vec{r}') \longrightarrow \delta(\vec{r}-\vec{r}')$，则

$D = \varepsilon E$，$B = \mu H$

我们将考虑 ε, μ 为复数时的物理

离开波源 $\vec{J}_f = 0$，$\rho_f = 0$

$$\nabla \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{E} = 0 .$$

在波动问题中，最直截了当的办法是根据对称性假定出波动的形式，然后找出色散（ω与k的关系）

比如这里：

振幅，为一矢量（包括丰富信息）

$$\vec{E}(\vec{r},t) = \boxed{\vec{\mathcal{E}}_0} e^{i\vec{k}\cdot\vec{r} - i\omega t} \Longrightarrow \text{物理场取实部.}$$

$$\vec{B}(\vec{r},t) = \boxed{\vec{B}_0} e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

$$\Longrightarrow i\vec{k} \times \vec{B}_0 = \mu\varepsilon(-i\omega)\vec{\mathcal{E}}_0 \Longrightarrow \vec{k} \times \vec{B}_0 = -\omega\mu\varepsilon\,\vec{\mathcal{E}}_0 \quad ①$$

$$i\vec{k} \times \vec{E}_0 = -(-i\omega)\vec{B}_0 \qquad \vec{k} \times \vec{\mathcal{E}}_0 = \omega\vec{B}_0 \quad ②$$

$$\vec{k} \cdot \vec{B}_0 = 0 \qquad \vec{k} \cdot \vec{B}_0 = 0 \quad ③$$

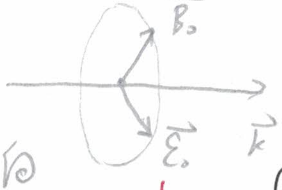
$$\vec{k} \cdot \vec{\mathcal{E}}_0 = 0 \qquad \vec{k} \cdot \vec{\mathcal{E}}_0 = 0 \quad ④$$

注：这里我们先考虑 μ, ε 均为实数，且 >0.

由③知 $\vec{B}_0 \perp \vec{k}$，

由④知 $\vec{\mathcal{E}}_0 \perp \vec{k}$，

$\Longrightarrow$ $\vec{B}_0$，$\vec{\mathcal{E}}_0$ 垂直于传播方向



↓

$\vec{B}_0$，$\vec{\mathcal{E}}_0$ 相对方向不清楚

$\vec{k} \times ①$ 知

$$\vec{k} \times (\vec{k} \times \vec{B_0}) = -\omega\mu\varepsilon\, \vec{k} \times \vec{\varepsilon_0} = -\omega^2\mu\varepsilon\, \vec{B_0}$$

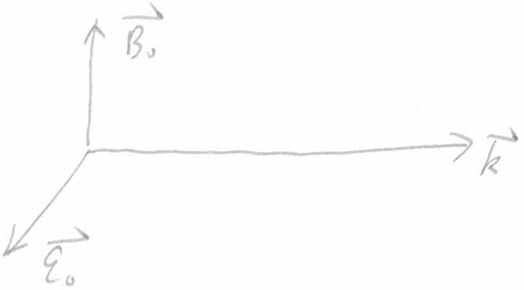
$$\downarrow$$

$$(\vec{k} \cdot \vec{B_0})\vec{k} - k^2\vec{B_0}$$

$\Longrightarrow$ $k^2 = \omega^2\mu\varepsilon$ $\Longrightarrow$ $\omega = \sqrt{\mu\varepsilon}\,|k|$，即推出色散关系

令 $\sqrt{\frac{1}{\mu\varepsilon}} = c$，$\omega = c|k|$，光的群速度 $\frac{\partial\omega}{\partial k} = c$

另外，由①，②可知 $\vec{\varepsilon_0}$，$\vec{B_0}$ 互相垂直



$\vec{\varepsilon_0}$，$\vec{B_0}$ 内部结构：

假定 $\vec{k}$ 沿z方向，$\vec{\varepsilon_0} = \varepsilon_{0x}\vec{x} + \varepsilon_{0y}\vec{y}$，$\boxed{\text{需保持垂直}}$

$\vec{B_0} = B_{0x}\vec{x} + B_{0y}\vec{y}$

$\boxed{\text{只需分析一个}}$

$\vec{\varepsilon_0} = \varepsilon_{0x}\vec{x} + \varepsilon_{0y}\vec{y}$

① ε_{0x}，ε_{0y} 均为实数，$\vec{\varepsilon_0}$ 沿某个固定方向，$\vec{E} = \vec{\varepsilon_0} e^{i\vec{k}\cdot\vec{r} - i\omega t}$

② ε_{0x}，ε_{0y} 均为虚数，与①并无大的区别

③ ε_{0x}，ε_{0y} 中一实一虚。

假定 $\vec{\varepsilon}_0 = \tilde{\varepsilon}_{0x}\vec{x} + i\tilde{\varepsilon}_{0y}\vec{y}$ ，“~”表示为实数

$$\vec{E} = \left(\tilde{\varepsilon}_{0x}\vec{x} + i\tilde{\varepsilon}_{0y}\vec{y}\right)e^{ik_z z - i\omega t}$$

物理场取实部

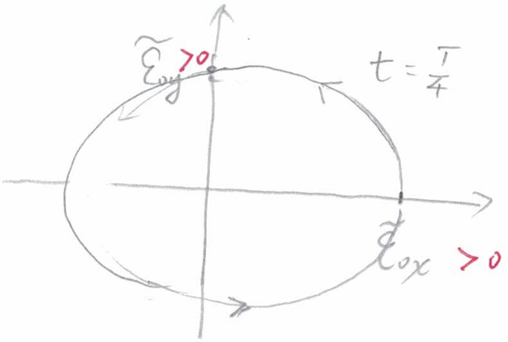
$$\mathrm{Re}\,E_x = \tilde{\varepsilon}_{0x}\cos(k_z z - \omega t)$$

$$\mathrm{Re}\,E_y = -\tilde{\varepsilon}_{0y}\sin(k_z z - \omega t)$$

我们找一个固定点，如 $z=0$，观看场如何随时间运动

$$\mathrm{Re}\,E_x(z=0) = \tilde{\varepsilon}_{0x}\cos(\omega t)$$

$$\mathrm{Re}\,E_y(z=0) = \tilde{\varepsilon}_{0y}\sin(\omega t)$$

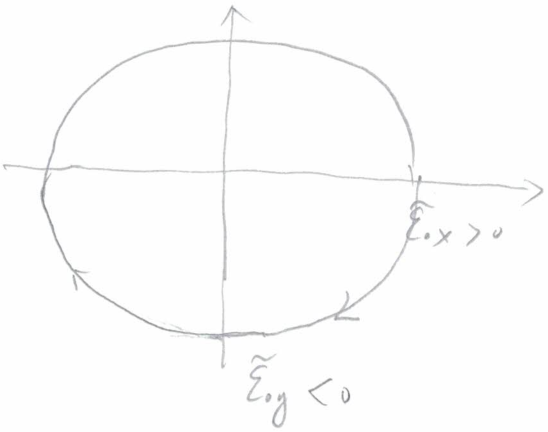


1° 根据 $\tilde{\varepsilon}_{0x}$，$\tilde{\varepsilon}_{0y}$ 相对符号，

有两种运动方式

① 逆时$\overset{\text{针}}{\text{刻}}$：右旋 ⇒ “右手法则”

② 顺时$\overset{\text{针}}{\text{刻}}$：左旋.



2°，$|\tilde{\varepsilon}_{0x}| = |\tilde{\varepsilon}_{0y}|$，圆偏振

① 逆时针：右旋圆偏振

② 顺时刻：左旋圆偏振

$$\varepsilon_{0x}\hat{\vec{x}} = \frac{\varepsilon_{0x}+i\varepsilon_{0x}}{2}\hat{\vec{x}}$$

$$〃 + \frac{\varepsilon_{0x}-i\varepsilon_{0x}}{2}\hat{\vec{y}}$$

3°. 沿固定方向运动 ⇒ 线偏振 ⟹ 总能分解成右旋，左旋圆偏振

有损介质的解

$$\begin{cases} \vec{k} \times \vec{B_0} = -\omega \mu \varepsilon \vec{\mathcal{E}_0} & ① \\ \vec{k} \times \vec{\mathcal{E}_0} = \omega \vec{B_0} & ② \\ \vec{k} \cdot \vec{B_0} = 0 & ③ \\ \vec{k} \cdot \vec{\mathcal{E}_0} = 0 & ④ \end{cases}$$

由②,

$$\vec{k} \times (\vec{k} \times \vec{\mathcal{E}_0}) = (\vec{k} \cdot \vec{\mathcal{E}_0})\vec{k} - \vec{k}^2 \vec{\mathcal{E}_0} = \omega \vec{k_0} \times \vec{B_0}$$

$$= -\omega^2 \mu \varepsilon \vec{\mathcal{E}_0}$$

$$\Longrightarrow \vec{k}^2 = \omega^2 \mu \varepsilon$$

$$(\vec{k}' + i\vec{k}'')^2 = \omega^2 (\mu' + i\mu'')(\varepsilon' + i\varepsilon'')$$

$$\vec{k}'^2 - k''^2 + 2i\vec{k}' \cdot \vec{k}'' = \omega^2 (\mu'\varepsilon' - \mu''\varepsilon'' + i\mu''\varepsilon' + i\mu'\varepsilon'')$$

$$\Longrightarrow k'^2 - k''^2 = \omega^2 (\mu'\varepsilon' - \mu''\varepsilon'')$$

$$2k'k'' = \omega^2 (\mu''\varepsilon' + \mu'\varepsilon'')$$

$$\Longrightarrow k'' = \omega \sqrt{\frac{(\varepsilon'\mu' - \varepsilon''\mu'')}{2} \left[\sqrt{1 + \frac{(\varepsilon''\mu' + \varepsilon'\mu'')^2}{(\varepsilon'\mu' - \varepsilon''\mu'')^2}} - 1 \right]}$$

$$k' = \omega \sqrt{\frac{(\varepsilon'\mu' - \varepsilon''\mu'')}{2} \left[\sqrt{1 + \frac{(\varepsilon''\mu' + \varepsilon'\mu'')^2}{(\varepsilon'\mu' - \varepsilon''\mu'')^2}} + 1 \right]}$$

$$\vec{E} = \vec{E_0} e^{-k'' \cancel{z} x} \cos(\omega t - k' z)$$

准静态问题 $\Longrightarrow$ 例：(电磁波在导体中的传播).

我们已经研究了 静止电荷产生的静电场
稳恒电流产生的静磁场

下面我们就要开始研究随时间变化的电磁场以及它们与介质的相互作用

我们要问：我们在静态问题得到的经验对我们研究含时问题有什么帮助？

如果场变化很慢，每时每刻我们还会可将场视为静态的.

Maxwell 方程组.

$$\begin{cases} \nabla\cdot\vec{D} = \rho_f \\ \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{H} = \underbrace{\vec{J}_f + \dfrac{\partial\vec{D}}{\partial t}}_{} \xrightarrow{} \text{辐射.} \end{cases}$$

这两项哪个大？(后期可能可以考虑辐射)

可以有频率依赖

$$\vec{J}_f = \sigma_c\vec{E}, \quad \vec{J}_D = \varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\vec{E} = \vec{E}_0 e^{-i\omega t} \Longrightarrow \vec{J}_D \sim \varepsilon(-i\omega\vec{E})$$

当 $\vec{J}_D \ll \vec{J}_f$ 时，$\varepsilon\omega \ll \sigma_c \Longrightarrow \omega \ll \frac{\sigma_c}{\varepsilon} = \omega_0$

例如：Cu： $\sigma_c \approx 5.9\times10^{7}\,\Omega^{-1}\cdot m^{-1}$,

$\Rightarrow \omega_\sigma \approx 6.7\times10^{18}$ Hz

可见光：~1000 THz → ~10^{15} Hz

一般对于宏观大的物体，可以暂时忽略辐射，

但是对于纳米光学，需要慎重，具体还需要分析尺寸的条件（可参考朗道的书）

对于均匀导体，没有电荷的堆积，$\rho_f = 0$.

（怎么 argue 这个事情？）

⇒ 电荷参与的情况

等离子体.

$$\frac{\partial\rho_f}{\partial t} = -\nabla\cdot\vec{J}_f = -\sigma\nabla\cdot\vec{E} = -\sigma\frac{\rho_f}{\varepsilon}$$

$$\rightarrow \rho_f(t)=\rho_0 e^{-\underbrace{\boxed{\frac{\sigma}{\varepsilon}}}_{\omega_\sigma}t}$$

⇒ 只要产生电荷堆积，电荷就会在时间尺度

$\frac{1}{\omega_\sigma} = \frac{\varepsilon}{\sigma}$ 衰减掉（以辐射的方式吗？）

（这个事情应该不是辐射，因为电荷是无法自己变成电磁波吗？似应启发讨论）

⇒ 只要电磁波的特征变化时间尺度够长，$\frac{1}{\omega} \gg \frac{1}{\omega_\sigma}$,

即 $\omega \ll \omega_\sigma$，可以认为没有电荷堆积.

Maxwell 方程组

$$\nabla\cdot\vec{D}=\rho$$

$$\nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

$$\vec{D}=\varepsilon_0\vec{E}+\vec{P}$$

$$\vec{H}=\frac{1}{\mu_0}\vec{B}-\vec{M}$$

$$\nabla\cdot\vec{P}=-\rho$$

$$\nabla\cdot\vec{J}=-\frac{\partial\rho}{\partial t} \Longrightarrow \vec{J}=\frac{\partial\vec{P}}{\partial t}$$

$$\vec{J}=\sigma\vec{E}$$

①我们考虑的都是线性材料，而且处理问题的尺度均大

材料本身的晶格长度，所以可以认为材料是均匀

② 响应是非局域的且具有频率依赖性

$$\vec{D}(\vec{k},\omega)=\varepsilon_0\,\varepsilon(\vec{k},\omega)\,\vec{E}(\vec{k},\omega)$$

$$\vec{J}(\vec{k},\omega)=\sigma(\vec{k},\omega)\,\vec{E}(\vec{k},\omega)$$

$$\varepsilon(\vec{k},\omega)=1+\frac{i\sigma(\vec{k},\omega)}{\varepsilon_0\omega}$$

推导：

$$\varepsilon_0\,\varepsilon(\vec{k},\omega)\,\vec{E}(\vec{k},\omega)=\varepsilon_0\vec{E}(\vec{k},\omega)+\vec{P}(\vec{k},\omega)$$

$$\Longrightarrow \vec{P}(\vec{k},\omega)=\varepsilon_0\left(\varepsilon(\vec{k},\omega)-1\right)\vec{E}(\vec{k},\omega)$$

$$\vec{J}(\vec{k},\omega)=-i\omega\vec{P}(\vec{k},\omega)$$

$$=-i\omega\,\varepsilon_0\left(\varepsilon(\vec{k},\omega)-1\right)\vec{E}(\vec{k},\omega)$$

$$=\sigma(\vec{k},\omega)\,\vec{E}(\vec{k},\omega)$$

$$\Longrightarrow \varepsilon(\vec{k},\omega)-1=\frac{i\sigma(\vec{k},\omega)}{\varepsilon_0\omega} \Longrightarrow \varepsilon(\vec{k},\omega)=1+\frac{i\sigma(\vec{k},\omega)}{\varepsilon_0\omega}$$

如果波长明显大于金属的特征长度（如电子的平均自由程），金属对光波介电常数只考虑对频率的依赖性

即 $\varepsilon(\vec{k}=0,\omega)=\varepsilon(\omega)$

2° 当金属的结构单元小于电子的平均自由程时，比如一些极小尺寸的金属针尖，就要考虑介电函数对空间位置的色散关系.

这里暂时考虑 case 1°

$$\varepsilon(\omega)=\varepsilon_1(\omega)+i\varepsilon_2(\omega)$$

$$\sigma(\omega)=\sigma_1(\omega)+i\sigma_2(\omega)$$

⟹ 电导率的实部对应介电函数虚部 ⟹ 代表吸收

电导率的虚部对应介电函数实部 ⟹ 极化强度大小

若没有外界的激励，Maxwell 方程具有行波形式

$$\nabla\times\nabla\times\vec{E}=-\mu_0\frac{\partial^2\vec{D}}{\partial t^2}$$

$$\hookrightarrow \nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}$$

$$\Longrightarrow \vec{k}(\vec{k}\cdot\vec{E})-k^2\vec{E}=-\varepsilon(\vec{k},\omega)\frac{\omega^2}{c^2}\vec{E}$$

(1). 横波 $\vec{k}\cdot\vec{E}=0$

$$\Longrightarrow k^2=\varepsilon(\vec{k},\omega)\frac{\omega^2}{c^2}$$

(2). 纵波 $\vec{k}\cdot\vec{E}=kE$ ⟹ $\varepsilon(\vec{k},\omega)=0$ ⟹ 电子振荡为集体振荡，对应金属中等离激元激发

$\vec{E}$ 为纯极化场

⇑

$\vec{D}=0$

⇑

金属介电函数表达式

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega} = 1 - \frac{\omega_p^2}{\omega^2 + \frac{i\omega}{\tau}}$$

$$= 1 - \frac{\omega_p^2}{\omega^2} \frac{1}{1 + i/\omega\tau}$$

$$= 1 - \frac{\omega_p^2}{\omega^2} \frac{1 - i/\omega\tau}{1 + \frac{1}{\omega^2\tau^2}}$$

$$= 1 - \omega_p^2 \frac{\tau^2 - \frac{i}{\omega}\tau}{\omega^2\tau^2 + 1}$$

$$\varepsilon_1(\omega) = 1 - \frac{\omega_p^2\tau^2}{1 + \omega^2\tau^2}$$

✓

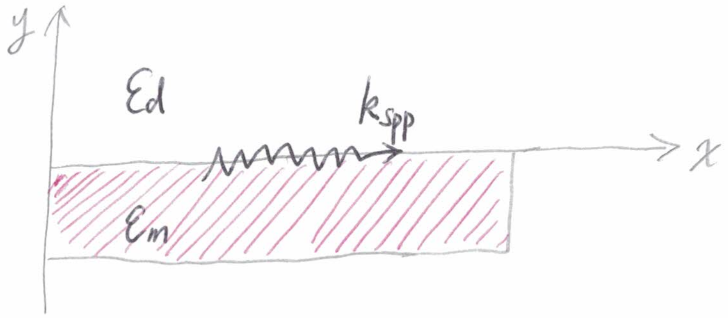
$$\varepsilon_2(\omega) = 1 + \frac{\frac{\omega_p^2}{\omega}\tau}{1 + \omega^2\tau^2}$$

当 $\omega > \omega_p$ 时，当 ω 很大，$\omega\tau \gg 1$，$|\varepsilon_1(\omega)| \gg |\varepsilon_2(\omega)|$

$\Rightarrow \varepsilon = 1 - \frac{\omega_p^2}{\omega^2} \Rightarrow k^2 = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\frac{\omega^2}{c^2}$ （核心）

$\Rightarrow k^2 = \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2} \Rightarrow \omega^2 = \omega_p^2 + k^2c^2$

表面等离子激元（Spp）是电磁波和金属表面的电子耦合，电子在金属/电介质表面上作集体振荡，它是一种表面波，其能量是沿着金属的表面传播，垂直于金属表面的方向能量是指数衰减的.



对于 TM 偏振（TE 偏振不被支持）

磁场垂直于 xy-平面

$$\boxed{\begin{aligned} H_z^d &= A\exp(-\alpha_d y)e^{i(kx-\omega t)} \\ H_z^m &= A\exp(-\alpha_m y)e^{i(kx-\omega t)} \end{aligned}}$$

SSP 求解:

$$\nabla\times\vec{E} = i\omega\vec{B}, \quad \nabla\times\vec{H} = -i\omega\vec{D}$$

$$\vec{D} = \epsilon_0\epsilon_r\vec{E}$$

$$\vec{B} = \mu_0\vec{H}$$

TM → "p" polarizati

$\vec{E}_m$, $\vec{E}_d$, $\vec{H}_m$, $\vec{H}_d$ 均要求出来

$$\Rightarrow \underline{\nabla\times\vec{E} = i\omega\mu_0\vec{H}}$$

$$\nabla\times\vec{H} = -i\omega\epsilon_0\epsilon_r\vec{E}$$

$$\nabla\times(\nabla\times\vec{H}) = -i\omega\epsilon_0\epsilon_r\underline{\nabla\times\vec{E}}$$

$$= \omega^2\epsilon_0\mu_0\epsilon_r\vec{H}$$

$$\Rightarrow \nabla\times(\nabla\times\vec{H}) = \omega^2\epsilon_0\mu_0\epsilon_r\vec{H}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & E_z \end{vmatrix} = \frac{\partial E_z}{\partial y}\vec{i} + \frac{\partial E_y}{\partial x}\vec{k} - \frac{\partial E_z}{\partial x}\vec{j} - \cancel{\frac{\partial E_y}{\partial z}}\vec{i}$$

$$\Rightarrow H_z = 0, \ H_y = 0.$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \boxed{\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\vec{i}} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\vec{k}$$

$$+\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\vec{j} \sim \underline{E_y\vec{j} + E_z\vec{k}}$$

⑮

我们只需要考虑 TM 模: $\begin{cases} \vec{E} = E_x \hat{x} + E_y \hat{y} \\ \vec{H} = H_z \hat{z} \end{cases}$

$$\boxed{\nabla \times (\nabla \times \vec{H}) = \omega^2 \epsilon_0 \mu_0 \epsilon_r \vec{H} = \epsilon_r \frac{\omega^2}{c^2} \vec{H}}$$

$$\boxed{\vec{H} = H_z(x,y)\vec{z}}$$

$$\rightarrow \nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H} = \epsilon_r \frac{\omega^2}{c^2} H$$

$$\rightarrow -\frac{\partial^2 \vec{H}}{\partial x^2} - \frac{\partial^2 \vec{H}}{\partial y^2} - \epsilon_r \frac{\omega^2}{c^2} \vec{H} = 0$$

$$\rightarrow -\frac{\partial^2 H_z}{\partial x^2} - \frac{\partial^2 H_z}{\partial y^2} - \epsilon_r \frac{\omega^2}{c^2} H_z = 0$$

$$\rightarrow k_x^2 H_z - \frac{\partial^2 H_z}{\partial y^2} - \epsilon_r k_0^2 H_z = 0$$

$$\rightarrow \frac{\partial^2 H_z}{\partial y^2} + (\epsilon_r k_0^2 - k_x^2) H_z = 0$$

$$H_z(x,y) = \begin{cases} A e^{ik_x x} e^{-\sqrt{k_x^2 - \epsilon_d k_0^2}\, y}, & y > 0 \\ A e^{ik_x x} e^{-\sqrt{k_x^2 - \epsilon_m k_0^2}\, y}, & y \leq 0 \end{cases}$$

如何求出 dispersion? (电场的连续性条件)

$$-i\omega \varepsilon_0 \varepsilon_r \vec{E} = \nabla \times \vec{H} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z \end{vmatrix} = \frac{\partial H_z}{\partial y} \vec{x} - \frac{\partial H_z}{\partial x} \vec{y}$$

$$E_x = \frac{i}{\omega \varepsilon_0 \varepsilon_r} \frac{\partial H_z}{\partial y} = - \frac{i}{\omega_0 \varepsilon_0 \varepsilon_r} \sqrt{k_x^2 - \varepsilon_r k_0^2}\, H_z$$

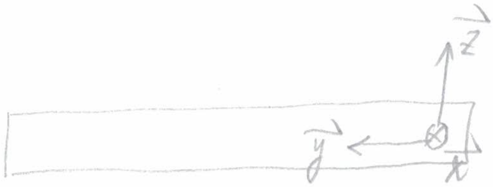
$$E_y = - \frac{i}{\omega \varepsilon_0 \varepsilon_r} \frac{\partial H_z}{\partial x} = \frac{k_x}{\omega_0 \varepsilon_0 \varepsilon_r} H_z$$

$$\Longrightarrow \quad \frac{i E_x}{\sqrt{k_x^2 - \varepsilon_r k_0^2}} = \frac{E_y}{k_x} \qquad \cancel{\boxed{m_y = i m_x}}$$

$$\Longrightarrow \quad \boxed{E_x = -i \frac{\sqrt{k_x^2 - \varepsilon_r k_0^2}}{k_x} E_y}$$

简单看一下 PRL $\underline{126}$, 257201 (2021)

$$\omega_a = v_{DMI} \cos\alpha \, k_y$$



感觉问题不是很大

The DMI between two atomic spins $\vec{S_i}$ and $\vec{S_j}$

$$H_{DMI} = - \vec{D_{ij}} \cdot (\vec{S_i} \times \vec{S_j})$$

$\boxed{\text{Chiral attenuation}}$

$kd \sim 1$, $\frac{2\pi}{k} = \lambda = 2\pi d \overset{\gg}{\ll} \lambda_{ex} \Rightarrow d \overset{\gg}{\ll} \frac{\lambda_{ex}}{2\pi}$

Normalization

$m_y = i m_x$, right circularly polarized

Near field

$$E_x^m = E_x^d$$

$$\Rightarrow -\frac{i}{\omega \varepsilon_0 \varepsilon_m} \sqrt{k_x^2 - \varepsilon_m k_0^2} = -\frac{i}{\omega \varepsilon_0 \varepsilon_d} \sqrt{k_x^2 - \varepsilon_d^2 k_0^2}$$

$$\Rightarrow \boxed{\frac{\sqrt{k_x^2 - \varepsilon_m k_0^2}}{\varepsilon_m} = \frac{\sqrt{k_x^2 - \varepsilon_d k_0^2}}{\varepsilon_d}}$$

$$\frac{\varepsilon_m^2}{\varepsilon_d^2} = \frac{k_x^2 - \varepsilon_m k_0^2}{k_x^2 - \varepsilon_d k_0^2}$$

ω_p

$$\varepsilon_m^2 (k_x^2 - \varepsilon_d k_0^2) = \varepsilon_d^2 (k_x^2 - \varepsilon_m k_0^2),$$

$$\Rightarrow \varepsilon_m^2 k_x^2 - \varepsilon_m^2 \varepsilon_d k_0^2 = \varepsilon_d^2 k_x^2 - \varepsilon_d^2 \varepsilon_m k_0^2$$

$$\Rightarrow (\varepsilon_d^2 \varepsilon_m - \varepsilon_m^2 \varepsilon_d) k_0^2 = (\varepsilon_d^2 - \varepsilon_m^2) k_x^2$$

$$\Rightarrow \varepsilon_d \varepsilon_m (\varepsilon_d - \varepsilon_m) k_0^2 = (\varepsilon_d - \varepsilon_m)(\varepsilon_d + \varepsilon_m) k_x^2$$

$$\Rightarrow \varepsilon_d \varepsilon_m k_0^2 = (\varepsilon_d + \varepsilon_m) k_x^2 \rightarrow k_0^2 = \frac{\varepsilon_d + \varepsilon_m}{\varepsilon_d \varepsilon_m} k_x^2$$

$$\Rightarrow \varepsilon_d \varepsilon_m \left(\frac{\omega}{c}\right)^2 = (\varepsilon_d + \varepsilon_m) k_x^2$$

$$\Rightarrow \omega = \sqrt{\frac{\varepsilon_d + \varepsilon_m(\omega)}{\varepsilon_d \varepsilon_m(\omega)}}\, c|k_x|$$

即 plasmon frequency

$$\varepsilon_m^{(\omega)} = 1 - \frac{\omega_p^2}{\omega^2}$$

when $\omega > \omega_p$, $\varepsilon_m^{(\omega)} > 0$, $\omega \rightarrow c|k_x|$ if $\omega \gg \omega_p$

For decay wave, $k_x^2 - \varepsilon_r k_0^2 > 0 \Rightarrow \varepsilon_r k_0^2 < k_x^2$

$$\Rightarrow \varepsilon_r \frac{\varepsilon_d + \varepsilon_m}{\varepsilon_d \varepsilon_m} < 1$$

$$\Rightarrow \begin{cases} \dfrac{\varepsilon_d + \varepsilon_m}{\varepsilon_m} < 1 \\ \dfrac{\varepsilon_d + \varepsilon_m}{\varepsilon_d} < 1 \end{cases} \Rightarrow \boxed{\varepsilon_m < 0} \Rightarrow \omega < \omega_p$$

$$\varepsilon_m(\omega_{spp}) + \varepsilon_d < 0 \Rightarrow 1 - \frac{\omega_p^2}{\omega^2} + \varepsilon_d < 0 \Rightarrow \frac{\omega_p^2}{\omega^2} > 1 + \varepsilon_d \Rightarrow \omega <$$

Pitaevski $\quad \vec{M} = -i f \vec{D} \times \vec{D}^{*} / (16\pi)$

Riccardo Hertel

$$\underset{\downarrow}{\frac{\partial}{\partial t} n} + \nabla (n \underset{\searrow}{\vec{v}}) = 0$$

electron density $\qquad$ electron velocity

electric current density

$$\vec{j} = e n \vec{v}$$

electric field $\quad \delta \vec{E}(\vec{r}, t) = \vec{E} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$

Response:

$$n \longrightarrow \langle n \rangle + \delta n$$

$$\vec{v} \longrightarrow \langle \vec{v} \rangle + \delta \vec{v}$$

$$\langle \delta n \rangle, \; \langle \delta \vec{v} \rangle = 0$$

$\Longrightarrow$ continuity equation

$$\frac{\partial}{\partial t} \langle n \rangle + \nabla \left(\langle n \rangle \langle \vec{v} \rangle + \langle \delta n \, \delta \vec{v} \rangle \right) = 0 \qquad ①$$

$$\frac{\partial}{\partial t} \delta n + \nabla \left(\langle n \rangle \delta \vec{v} \right) = 0 \qquad ②$$

$$\delta \vec{j} = \sigma \, \delta \vec{E} = e \langle n \rangle \delta \vec{v} \qquad ③$$

$$\Longrightarrow \quad \delta \vec{v} = \frac{\sigma \, \delta \vec{E}}{e \langle n \rangle}$$

#②

$$-i\omega\,\delta n + \nabla\left(\langle n\rangle \frac{\sigma\,\delta\vec{E}}{e\langle n\rangle}\right) = 0$$

$$\Longrightarrow\ -i\omega\,\delta n + \nabla\left(\frac{\sigma}{e}\,\delta\vec{E}\right) = 0$$

$$\Longrightarrow\ \boxed{\delta n = -\frac{i}{\omega e}\nabla(\sigma\,\delta\vec{E})}$$

$$\delta\vec{j} = e\langle n\rangle\,\delta\vec{v} = \sigma\,\delta\vec{E}$$

$$\Longrightarrow\ \boxed{\delta\vec{v} = \frac{\sigma}{\langle n\rangle e}\,\delta\vec{E}}$$

$$\langle\vec{j}\rangle = \langle \delta n\,\delta\vec{v}\rangle$$

$$= \frac{1}{4}\left[\left(-\frac{i}{\omega e}\nabla(\sigma\,\delta\vec{E})\right)\frac{\sigma^*}{\langle n\rangle e}\,\delta\vec{E}^* + \left(\frac{i}{\omega e}\nabla(\sigma^*\delta\vec{E}^*)\right)\frac{\sigma}{\langle n\rangle e}\,\delta\vec{E}\right]$$

$$= -\frac{i}{4\langle n\rangle e^2\omega}\left[\nabla(\sigma\,\delta\vec{E})\,\sigma^*\,\delta\vec{E}^* - \nabla(\sigma^*\,\delta\vec{E}^*)\,\delta\vec{E}\right]$$

$$= -\frac{i}{4e\langle n\rangle\omega}\nabla\times\left[\sigma^*\vec{E}^*\times\sigma\vec{E}\right] + \vec{P}$$

㊶′

$\vec{P}$ ↓ ponderomotive force

$$\Longrightarrow\ \vec{M} = -\frac{i}{4e\langle n\rangle\omega}\,\sigma^*\vec{E}^*\times\sigma\vec{E}\ ,\ \sigma = \frac{i\langle n\rangle e^2}{\ }\ \ \omega_p = \sqrt{\frac{\langle n\rangle e^2}{\ }}$$

$$\vec{M} = -\frac{i}{4e\langle n\rangle\omega}\,\alpha^*\vec{E}^*\times\alpha\vec{E}$$

$$= -\frac{i}{4e\langle n\rangle\omega}\,\frac{\langle n\rangle^2 e^4}{m^2\omega^2}\,\vec{E}^*\times\vec{E}$$

$$= -\frac{i\langle n\rangle e^3}{4m^2\omega^3}\,\vec{E}^*\times\vec{E}$$

$$= -\frac{i\langle n\rangle e^3}{4m^2\omega^3}\,\frac{m\varepsilon_0}{\langle n\rangle e^2}\,\omega_p^2\,\vec{E}^*\times\vec{E}$$

$$= -\frac{ie\varepsilon_0}{4m\omega^3}\,\omega_p^2\,\vec{E}^*\times\vec{E}$$

$$= \boxed{\frac{ie\varepsilon_0}{4m\omega^3}\,\omega_p^2\,\vec{E}\times\vec{E}^*}$$

Oue 等人推的结果.

$$\vec{M} = -\frac{1}{2\omega}\,\frac{e\omega}{4m}\varepsilon_0\frac{d\epsilon}{d\omega}\,\mathrm{Im}\,(\vec{E}^*\times\vec{E})$$

$$\epsilon = 1-\frac{\omega_p^2}{\omega^2}$$

$$\Longrightarrow \frac{d\epsilon}{d\omega} = 2\,\frac{\omega_p^2}{\omega^3}$$

旋磁比 $-\frac{e}{2m}$

$$= -\frac{1}{2}\,\frac{e}{4mc}\varepsilon_0\,2\,\frac{\omega_p^2}{\omega^3}\,\mathrm{Im}\,(\vec{E}^*\times\vec{E})$$

$$= -\frac{e\,\omega_p^2\varepsilon_0}{4m\omega^3}\,\mathrm{Im}\,(\vec{E}^*\times\vec{E})$$ ✓

我们得到准静态条件下的 Maxwell 方程组

$$\begin{cases} \nabla\cdot\vec{E}=0 \\ \nabla\times\vec{E}=-\mu\dfrac{\partial\vec{H}}{\partial t} \\ \nabla\cdot\vec{H}=0 \\ \nabla\times\vec{H}=\sigma_c\vec{E} \end{cases} \Longrightarrow \nabla\times\nabla\times\vec{E}=-\mu\frac{\partial\nabla\times\vec{H}}{\partial t}=-\mu\sigma_c\frac{\partial\vec{E}}{\partial t}$$

$$\downarrow$$

$$\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}$$

$$\Longrightarrow \frac{\partial\vec{E}}{\partial t}=\frac{1}{\mu\sigma_c}\nabla^2\vec{E}$$

$$\frac{\partial\vec{H}}{\partial t}=\frac{1}{\mu\sigma_c}\nabla^2\vec{H}$$

⟹ 这类方程称为

扩散方程

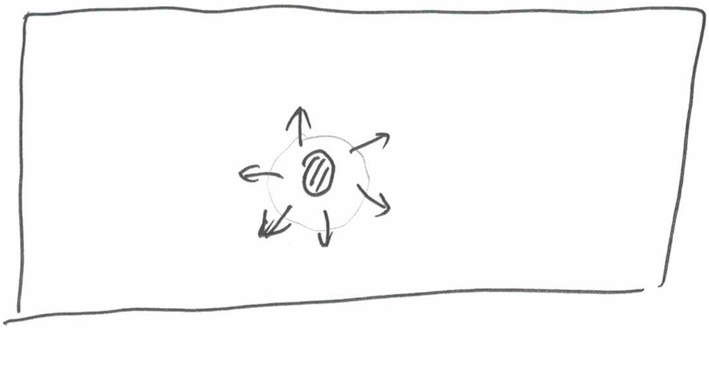
$\boxed{D\equiv\frac{1}{\mu\sigma_c}\ \text{可以称为扩散系数}}$

σ_c 越大，D 越小，扩散越慢。理想导体 $\sigma_c\to\infty$，$D\to 0$，完全不扩散。在绝缘体中，$\sigma_c\to 0$，D 非常大，电磁场很容易扩散 → 有些悖论？

（原因：导体中能用电荷阻碍电磁场的扩散？）

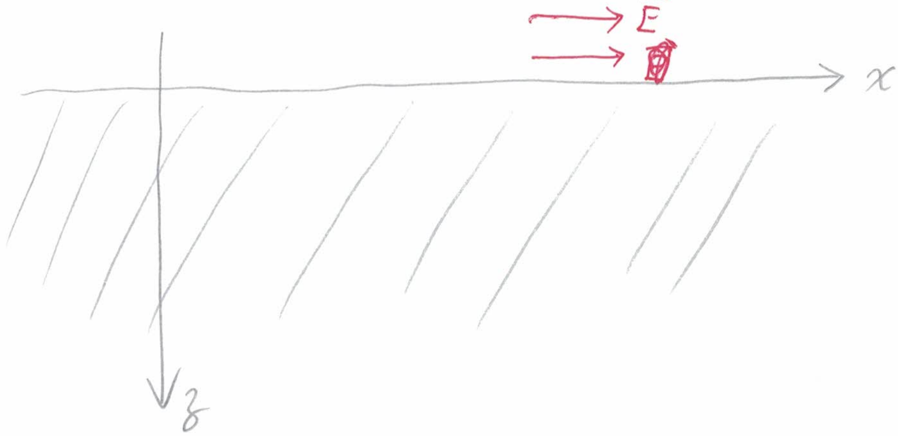
⇓

电荷的作用如何体现出来？



例:

电磁波在导体中的传播



考虑导体占据 $z>0$ 的半个空间，一个沿 x 方向在 x-y 平面均匀的谐变电场驱动导体电流沿 x 方向运动

$$\vec{E} = \mathcal{E}_x(z)\, e^{-i\omega t}\, \vec{x}$$

代入到扩散方程中 $\left(\frac{\partial^2}{\partial z^2} + i\mu\sigma_c\omega\right)\mathcal{E}_x(z) = 0$

$$\mathcal{E}(z) = E_0 e^{-pz} \Longrightarrow p = \pm\sqrt{\frac{1}{2}\omega\mu\sigma_c}\,(1-i) = \pm\alpha(1-i)$$

其中 $\alpha = \sqrt{\frac{1}{2}\omega\mu\sigma_c}$

$$\vec{E} = \vec{x}\left(E_0 e^{-\alpha(1-i)z} + \cancel{E_0' e^{\alpha(1-i)z}}\right) e^{-i\omega t}$$

$$\Longrightarrow \vec{E} = \vec{x} E_0 e^{-\alpha z} \cos(\omega t - \alpha z)$$

$\Longrightarrow$ 远离导体表面，电磁场振荡衰减，其衰减的特征长度为趋肤深度

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\mu\omega\sigma_c}}$$

~~准静态电磁场~~

eddy current：能量在导体中如何耗散？

$\omega = ck \rightarrow k = \frac{\omega}{c} \rightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega}$

Have a closer look at Maxwell equations

$\nabla \cdot \vec{E} = 0 \quad , \quad \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$

$\nabla \cdot \vec{H} = 0 \quad , \quad \nabla \times \vec{H} = \sigma_c \vec{E}$

边界条件：

$\vec{B}_n$ 连续， $\vec{H}_t$ 连续

$\nabla \cdot \vec{J} = 0 \Longrightarrow$ 在导体表面 $J_n = 0$，即 $\vec{E}_n = 0$.

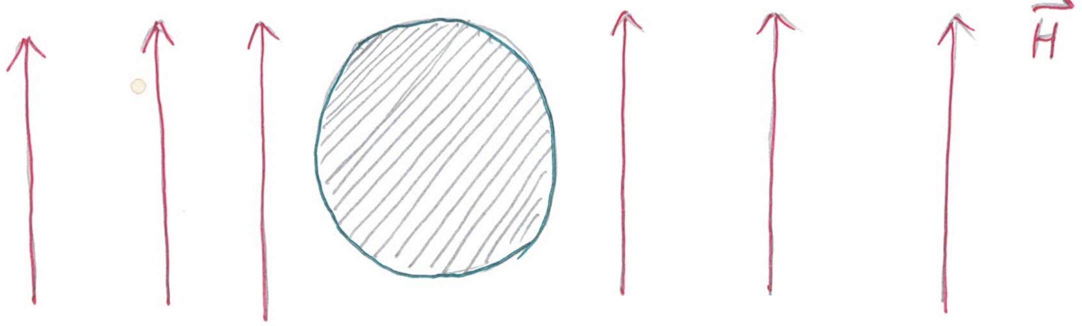
现在将一块导体放入到频率为 ω 的交变磁场中

$$\frac{\partial \vec{H}}{\partial t} = \frac{1}{\mu \sigma_c} \nabla^2 \vec{H}$$

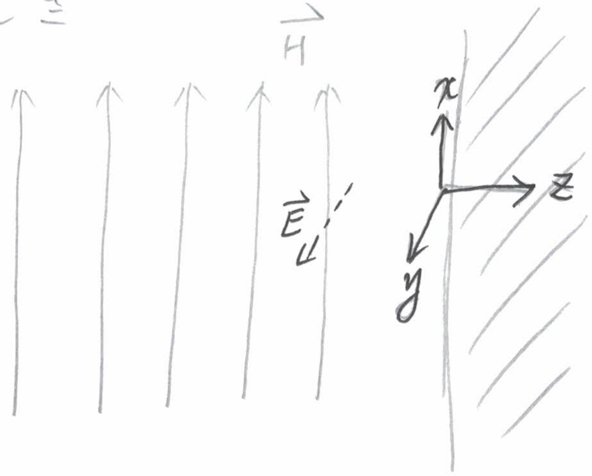
1) $\delta \gg l$， $\vec{H}$ 均匀

2) $\delta \ll l$， $\vec{H}$ 如何分布，$\vec{E}$ 如何分布

$\vec{H} \approx \vec{H}_0 e^{-i\omega t} e^{i\vec{k}\cdot\vec{r}}$



简化些



半无限大的导体.

$$\vec{H}(z) = \vec{H}_0(z)\, e^{-i\omega t}\, \vec{x}$$

$$\vec{H} = \vec{x}\, H_0\, e^{-\alpha z} \cos(\omega t - \alpha z)$$

$$\vec{E} = \frac{1}{\sigma_c} \nabla \times \vec{H} = \frac{1}{\sigma_c} \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ 0 & 0 & \frac{\partial}{\partial z} \\ H_x & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{\sigma_c} \frac{\partial H_x}{\partial z} \vec{y} = \frac{H_0}{\sigma_c} \frac{\partial}{\partial z} \left(e^{-\alpha z} \cos(\omega t - \alpha z) \right) \vec{y}$$

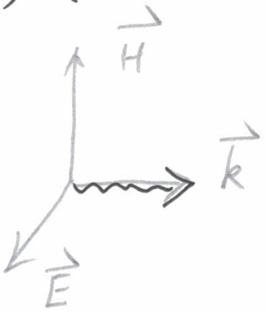
$$= \frac{H_0}{\sigma_c} \left(-\alpha e^{-\alpha z} \cos(\omega t - \alpha z) - \alpha e^{-\alpha z} \sin(\omega t - \alpha z) \right) \vec{y}$$

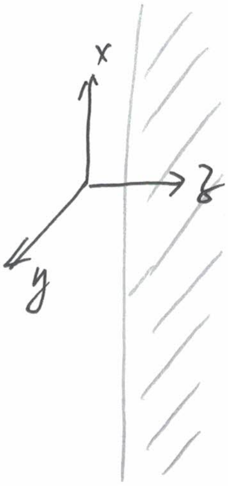
$$= -\frac{\alpha H_0}{\sigma_c} e^{-\alpha z} \left(\cos(\omega t - \alpha z) + \sin(\omega t - \alpha z) \right) \vec{y}$$

$$= -\frac{\alpha H_0}{\sigma_c} e^{-\alpha z} \sqrt{2} \cos\Big(\omega t - \alpha z \underset{\sim}{\overset{\text{相位差}}{-\frac{\pi}{4}}}\Big)\ \vec{y} \rightarrow \text{方向}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

波动：





$$\vec{H}(z) = H_0(z)\, e^{-i\omega t + ik_z z}\ \vec{x}$$

$$\vec{E}(z) = E_0(z)\, e^{-i\omega t + ik_z z}\ \vec{y} \quad \Longrightarrow \quad \underline{\vec{E} = -\frac{1}{\omega\varepsilon}\vec{k}\times\vec{H}}$$

不考虑辐射，就全部耗损在导体表面上.

$$\frac{\partial \vec{H}}{\partial t} = \frac{1}{\mu\sigma_c}\nabla^2\vec{H}$$

$$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\mu\sigma_c}\nabla^2\vec{E} \quad \Longrightarrow \quad -i\omega\begin{pmatrix}\vec{H}\\ \vec{E}\end{pmatrix} = \frac{1}{\mu\sigma_c}\frac{\partial^2}{\partial z^2}\begin{pmatrix}\vec{H}(z)\\ \vec{E}(z)\end{pmatrix}$$

$$\Longrightarrow \quad -i\omega\begin{pmatrix}\vec{H}_0(z)\\ \vec{E}_0(z)\end{pmatrix} = \frac{1}{\mu\sigma_c}\left(-k^2 + \frac{\partial^2}{\partial z^2}\right)\begin{pmatrix}\vec{H}_0(z)\\ \vec{E}_0(z)\end{pmatrix}$$

$$\Longrightarrow \quad \left(-i\omega\mu\sigma_c + k^2\right)\begin{pmatrix}\vec{H}_0(z)\\ \vec{E}_0(z)\end{pmatrix} = \frac{\partial^2}{\partial z^2}\begin{pmatrix}\vec{H}_0(z)\\ \vec{E}_0(z)\end{pmatrix}$$

$-i\omega_{eff}\mu\sigma_c$

$$e^{-pz}$$

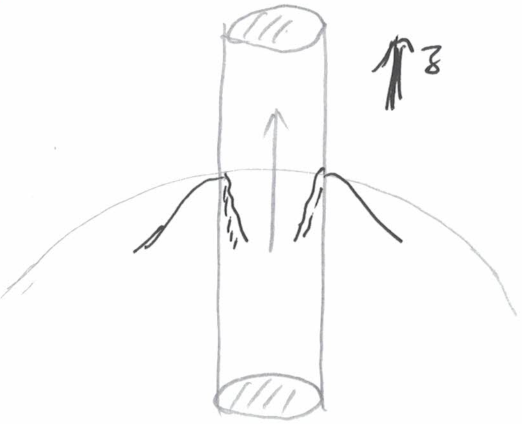
$\omega_{eff} = \omega + i\frac{k^2}{\mu\sigma_c}$,

$p^2 = \sqrt{-i\omega\mu\sigma_c + k^2}$ ✓

电磁波在导体中现在均是横波情况

纵波是否可以传播 | $\vec{k}\cdot\vec{E}_0 \neq 0$ 时.

我们研究下圆截面导线



$E_z(\rho)\,e^{-i\omega t}$

$$\frac{1}{\mu\sigma_c}\nabla^2\vec{E} = \frac{\partial\vec{E}}{\partial t}$$

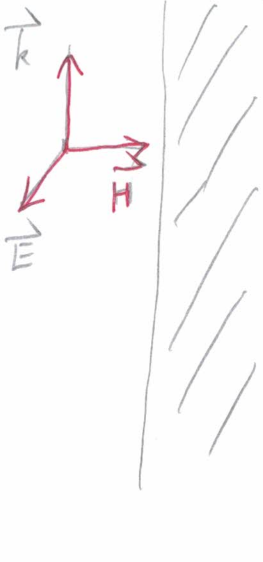
$$\to \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right) + k^2E_z = 0$$

$$\to E_z \propto J_0(kr)\,e^{-i\omega t}$$

↓贝塞尔函数

导线外呢? ⟹ 真空Maxwell方程组

$\nabla^2\vec{E} = -\frac{\omega^2}{c^2}\vec{E}$



问题: 解 $\vec{E}_\rho(\rho)\,e^{i\overset{\vec{k}\cdot\vec{r}}{k_z z}-i\omega t}$

是否允许?

$\nabla\cdot\vec{E} = 0$

电磁波在导线中如何传播

是否可以求出解来

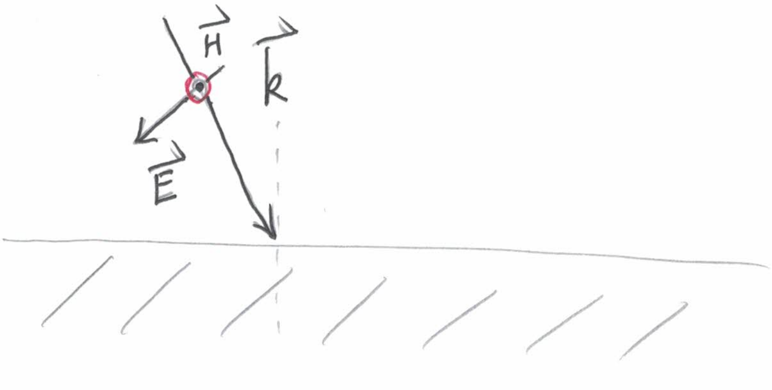
(看看准静态下是否可以求解?)

$$\frac{1}{\mu\sigma_c}\nabla^2\vec{E} = -i\omega\vec{E}$$

$E_n|_s = 0$

作业.

$\vec{E}_\parallel$ 连续



这个时候如何求解呢!

讨论：1). 对于Cu，特征频率为GHz（天线）的电磁波，这一深度为微米量级。

2). 纳米材料厚度远小于趋肤深度，可认为电磁场在其中均匀分布.

3). 电磁波"喜欢"沿导线表面传播。这不是没有必要用很厚的导线传播电磁波（交流电）？2也上.

a) 在高频电路中可用空心铜导线代替实心铜导线以节约铜材

b). 利用趋肤效应可对金属表面进行热处理（淬火），使表面性质与内部性质不同，如表面更耐磨、更坚硬，而内部更软等。

更基本的原因：ε 与 σ 的关系.

为什么内部无电场？

有耗介质：

$\vec{k} \times \vec{H_0} = \sigma \vec{E_0} - i\omega\varepsilon\vec{E_0}$

$\vec{k} \times \vec{E_0} = \mu\omega \vec{H_0}$

$\vec{k} \cdot \vec{H_0} = 0$

$\vec{k} \cdot \vec{E_0} = 0$

$\Longrightarrow$ σ 的作用，类似于 $\boxed{\varepsilon''}$

$\Downarrow$

有ε''时 $\vec{k} = \vec{k'} + i\vec{k''}$ 为复数.

$\varepsilon'' \rightarrow \frac{\sigma}{\omega}$ ~~θ~~, $\mu'' = 0$ $\sqrt{\frac{\omega\mu\sigma}{2}}$

$k'' = \omega \sqrt{\frac{\varepsilon\mu}{2}\left[\sqrt{1+\frac{(\frac{\sigma}{\omega}\mu^{\circ})^2}{\mu^{\circ}}} - 1\right]}$

单能流

$\vec{S} = \vec{E} \times \vec{H} \quad \propto \vec{k}$

各向异性介质中更有趣，可参见朗道连续介质力学.

电磁波在介质界面上的反射和折射

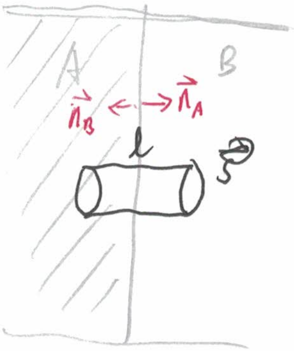
首先考虑普通电介质，不包括 $\vec{M}$, $\vec{P}$，也不为导体.

普通电介质为各向同性.

麦克斯韦方程组（暂时包括进自由电荷，自由电流）

其它情形应该学会推之思考

$$\boxed{\begin{aligned} &\nabla \cdot \vec{D} = \rho_f \quad ① \\ &\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ② \\ &\nabla \cdot \vec{B} = 0 \quad ③ \\ &\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \quad ④ \end{aligned}}$$



⟹ 如何导出边界条件

$\int \vec{D} \cdot d\vec{S}$

$\|$

$\int d\vec{S}\, dl\, \nabla \cdot \vec{D} = \int d\vec{S}\, dl\, \rho_f$

由 ①，$\cancel{\vec{D}_\perp}$ $\quad \vec{D}_\perp(A) - \vec{D}_\perp(B) = \sigma_f$ → 面电荷密度

由 ③，$\quad \vec{B}_\perp(A) = \vec{B}_\perp(B)$

由 ②，$\quad \vec{E}_\parallel(A) = \vec{E}_\parallel(B)$

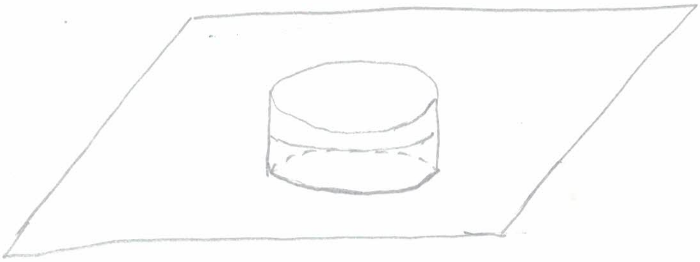
由 ④，$\quad \vec{H}_\parallel(A) - \vec{H}_\parallel(\vec{B}) = \overset{\vec{\alpha}_f}{\cancel{\vec{j}_f}}$ → 面电流密度

边界条件的推导:

积分形式的 Maxwell 方程组

$$\boxed{\begin{aligned}
&\oint_L \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S} && \text{①}\\
&\oint_L \vec{H}\cdot d\vec{l} = \int_\Sigma \vec{j}_f\cdot d\vec{S} + \frac{d}{dt}\int_\Sigma \vec{D}\cdot d\vec{S} && \text{②}\\
&\oint_S \vec{D}\cdot d\vec{S} = \int_V \rho_f \, dV && \text{③}\\
&\oint_S \vec{B}\cdot d\vec{S} = 0 && \text{④}
\end{aligned}}$$

对③考察下



侧面很小 $\Longrightarrow (D_{2n} - D_{1n})\Delta S = \left(\underline{\int dl\, \rho_f}\right)\Delta S = \sigma_f \Delta S$

$$\Longrightarrow \boxed{D_{2n} - D_{1n} = \sigma_f}$$

其它类似推导.

$$\boxed{\begin{aligned}&\vec{e}_n\times(\vec{E}_2-\vec{E}_1)=0\\&\vec{e}_n\times(\vec{H}_2-\vec{H}_1)=\vec{\alpha}_f\\&\vec{e}_n\cdot(\vec{D}_2-\vec{D}_1)=\sigma_f\\&\vec{e}_n\cdot(\vec{B}_2-\vec{B}_1)=0\end{aligned}}$$

~~绝缘介质 $\vec{\alpha}_f=0$, $\sigma_f=0$~~

单色光 ↓

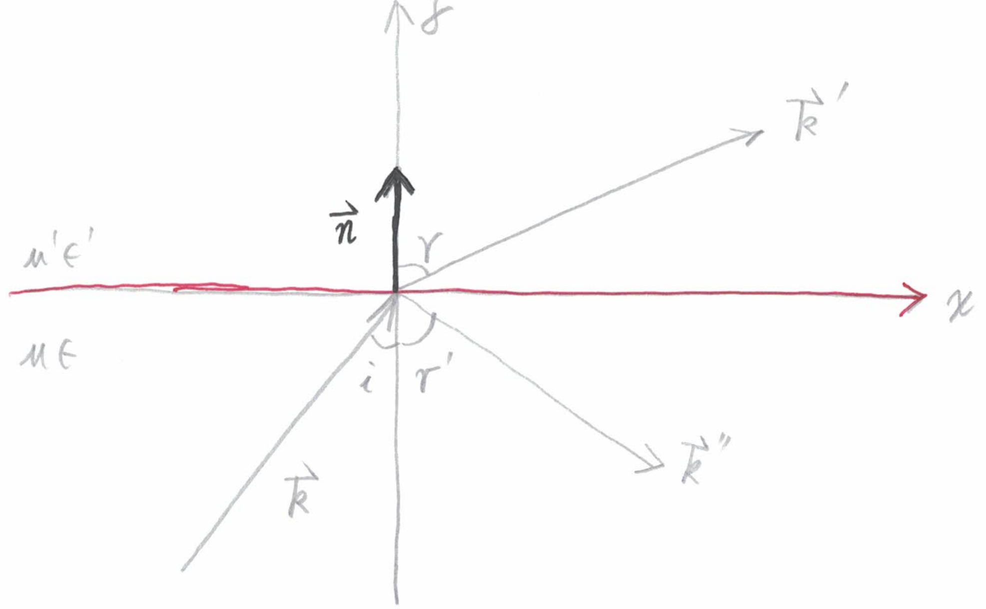
~~线~~性 均匀介质 $\vec{D}(\omega)=\varepsilon(\omega)\vec{E}(\omega)$, $\vec{B}(\omega)=\mu(\omega)\vec{H}(\omega)$

$$\Rightarrow\quad\begin{aligned}&\vec{e}_n\times\left(\vec{E}_2(\omega)-\vec{E}_1(\omega)\right)=0\\&\vec{e}_n\times\left(\vec{H}_2(\omega)-\vec{H}_1(\omega)\right)=\vec{\alpha}_f\\&\vec{e}_n\cdot\left(\varepsilon_2(\omega)\vec{E}_2(\omega)-\varepsilon_1(\omega)\vec{E}_1(\omega)\right)=\sigma_f\\&\vec{e}_n\cdot\left(\mu_2\vec{H}_2(\omega)-\mu_1\vec{H}_1(\omega)\right)=0\end{aligned}$$

在"1"和"2"介质内部, Maxwell 方程组分别满足

$$\begin{aligned}&\nabla\times\vec{E}_{(i)}=i\omega\mu_i\vec{H}_{(i)}\\&\nabla\times\vec{H}_{(i)}=-i\omega\varepsilon_i\vec{E}_{(i)}\\&\nabla\cdot\vec{E}_{(i)}=0\\&\nabla\cdot\vec{H}_{(i)}=0\end{aligned}$$

后面两式可由前面两式推出.



参考Jackson书

$\{\vec k, \vec k', \vec k''\}$ 是否在一个平面上，需要论证.

入射平面波: (incident)

$$\vec E = \vec E_0 e^{i\vec k\cdot\vec r - i\omega t}$$

$$\vec B = \sqrt{\mu\epsilon}\,\frac{\vec k\times\vec E}{k}$$

折射波: (refracted)

$$\vec E' = \vec E_0' e^{i\vec k'\cdot\vec r - i\omega t}$$

$$\vec B' = \sqrt{\mu'\epsilon'}\,\frac{\vec k'\times\vec E'}{k'}$$

反射波: (reflected)

$$\vec E'' = \vec E_0'' e^{i\vec k''\cdot\vec r - i\omega t}$$

$$\vec B'' = \sqrt{\mu\epsilon}\,\frac{\vec k''\times\vec E''}{k}$$

显然:

$|\vec k| = k = \omega\sqrt{\mu\epsilon}$

$|\vec k'| = k' = \omega\sqrt{\mu'\epsilon'}$

$|\vec k| = |\vec k''|$

这不是矢量方程.

$\vec r = x\vec x + y\vec y + z\vec z$, $z=0$ 时处于边界

边缘$\vec r$ = $\vec r$ $(z=0)$

$\rho_f = 0$，对于介电体.（后面也会考虑导体）

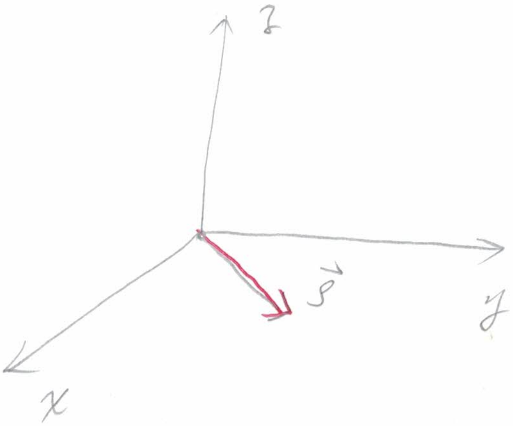
$\vec{D}_\perp$ 连续，$\vec{E}_\parallel$ 连续

$$\vec{E}_{0\parallel} e^{i\vec{k}\cdot\vec{\rho} - i\omega t} + \vec{E}'_{0\parallel} e^{i\vec{k}'\cdot\vec{\rho} - i\omega t}$$

$$= \vec{E}''_{0\parallel} e^{i\vec{k}''\cdot\vec{\rho} - i\omega t}$$

$$\Longrightarrow \vec{E}_{0\parallel} e^{i\vec{k}\cdot\vec{\rho}} + \vec{E}'_{0\parallel} e^{i\vec{k}'\cdot\vec{\rho}} = \vec{E}''_{0\parallel} e^{i\vec{k}''\cdot\vec{\rho}}$$

<u>对每一个 $\vec{\rho}$ 均成立</u>



$\Longrightarrow \vec{k}\cdot\vec{\rho} = \vec{k}'\cdot\vec{\rho} = \vec{k}''\cdot\vec{\rho}$

$\Longrightarrow$ ① $\vec{k}_\parallel = \vec{k}'_\parallel = \vec{k}''_\parallel$ ② 位于同一平面

$\Longrightarrow k\sin i = k'\sin r = k''\sin r'$

即折射，反射定律 $\Downarrow$ Snell's law

$|\vec{k}| = |\vec{k}''| = \omega\sqrt{\mu\epsilon}$

$|\vec{k}'| = \omega\sqrt{\mu'\epsilon'}$

振幅之间的关系 $\Longrightarrow$ 更多边界条件

$\vec{E}_\parallel$ 连续 ①
$D_\perp$ 连续（无 ρ_f）②
$\vec{H}_\parallel$ 连续（无 $\vec{\alpha}_f$）③
$\vec{B}_\perp$ 连续 ④.

<u>逐条写出</u>

① $(\vec{E}_0 + \vec{E}_0'' - \vec{E}_0') \times \vec{n} = 0$

② $\left[\epsilon (\vec{E}_0 + \vec{E}_0'') - \epsilon' \vec{E}_0' \right] \cdot \vec{n} = 0$

③ $$\left(\frac{1}{\mu} \sqrt{\mu\epsilon} \, \frac{\vec{k} \times \vec{E}_0}{k} + \frac{1}{\mu} \sqrt{\mu\epsilon} \, \frac{\vec{k}'' \times \vec{E}_0''}{k} - \frac{1}{\mu'} \sqrt{\mu'\epsilon'} \, \frac{\vec{k}' \times \vec{E}_0'}{k'} \right) \times \vec{n} = 0$$

两边同乘 ω，利用 $k = \omega\sqrt{\mu\epsilon}$，$k' = \omega\sqrt{\mu'\epsilon'}$ 可得

$$\left[\frac{1}{\mu} (\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0'') - \frac{1}{\mu'} (\vec{k}' \times \vec{E}_0') \right] \times \vec{n} = 0$$

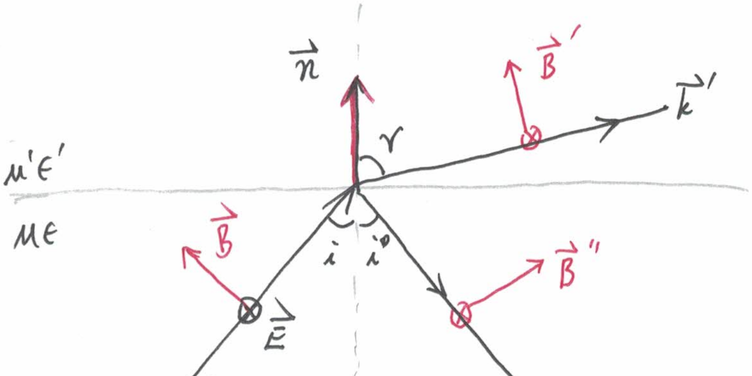
④ $(\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0'' - \vec{k}' \times \vec{E}_0') \cdot \vec{n} = 0$

考虑线偏振的横波，其它情形是以下两种情形的叠加

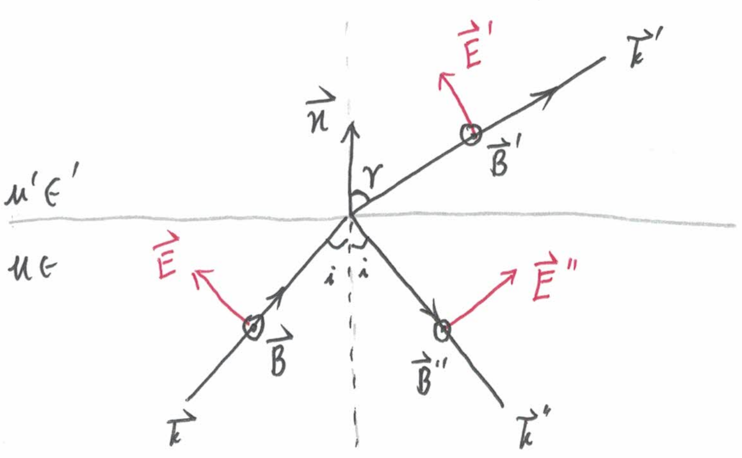
rotation direction

polarization direction

counterclockwise



(a)



(b)

情形 (a) 四个边界条件看看给出什么样的方程

① $E_0 + E_0'' - E_0' = 0$ ①′

② 直接满足

③ $\vec{k} \times \vec{E}$ 沿 $\vec{B}$ 方向✓，幅值为 kE

$$\vec{B} \times \vec{n} = B \sin(\vec{n}\text{与}\vec{B}\text{夹角}) \hookrightarrow \frac{\pi}{2} - i$$

$$= B \cos(\vec{n}\text{与}\vec{B}\text{夹角})$$

$$\Longrightarrow \frac{1}{\mu} k E_0 \cos i + \frac{1}{\mu} k E_0'' \cos i - \frac{1}{\mu'} k' E_0' \cos \gamma = 0$$

$$\Longrightarrow \sqrt{\frac{\epsilon}{\mu}} (E_0 - E_0'') \cos i - \sqrt{\frac{\epsilon'}{\mu'}} E_0' \cos r = 0 \quad ②'$$

①′与②′已经给出两个方程，可求出相对值

④应该为冗余的（作业：证明④为冗余并给出解释）

由①′可得

$$E_0' = E_0 + E_0''$$

代入②′中，$\sqrt{\frac{\epsilon}{\mu}} (E_0 - E_0'') \cos i = \sqrt{\frac{\epsilon'}{\mu'}} (E_0 + E_0'') \cos r$

$$\Longrightarrow \sqrt{\frac{\epsilon}{\mu}} E_0 \cos i - \sqrt{\frac{\epsilon'}{\mu'}} E_0 \cos r = \sqrt{\frac{\epsilon}{\mu}} E_0'' \cos i + \sqrt{\frac{\epsilon'}{\mu'}} E_0'' \cos r$$

$$\Longrightarrow \frac{E_0''}{E_0} = \frac{\sqrt{\frac{\epsilon}{\mu}} \cos i - \sqrt{\frac{\epsilon'}{\mu'}} \cos r}{\sqrt{\frac{\epsilon}{\mu}} \cos i + \sqrt{\frac{\epsilon'}{\mu'}} \cos r}$$

$$E_0' = E_0 + E_0''$$

$$= \Big(\frac{2\sqrt{\frac{\epsilon}{\mu}}\cos i}{\sqrt{\frac{\epsilon}{\mu}}\cos i + \sqrt{\frac{\epsilon'}{\mu'}}\cos r} E_0$$

$$\Longrightarrow \frac{E_0'}{E_0} = \frac{2\sqrt{\epsilon/\mu}\cos i}{\sqrt{\frac{\epsilon}{\mu}}\cos i + \sqrt{\frac{\epsilon'}{\mu'}}\cos r}$$

定义折射率. $n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$, $n' = \sqrt{\frac{\mu'\epsilon'}{\mu_0\epsilon_0}}$

$$\frac{E_0'}{E_0} = \frac{2\sqrt{\epsilon\mu}\cos i}{\sqrt{\epsilon\mu}\cos i + \frac{\mu}{\mu'}\sqrt{\epsilon'\mu'}\cos r}$$

$$= \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}n'\cos r}$$

$$= \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}n'\sqrt{1 - \sin^2 i\left(\frac{n}{n'}\right)^2}}$$

$$= \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

$$\frac{\sin i}{\sin r} = \frac{n'}{n}$$

也可得:

$$\frac{E_0''}{E_0} = \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

分析

1) n', n 可以为复数

2). 能量是否守恒?

$$(2n\cos i)^2 + \left(n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}\right)^2$$

$$\overset{?}{=} \left(n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}\right)^2$$

$$4n^2\cos^2 i - 4n\cos i\,\frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i} \overset{?}{=} 0$$

$$\Rightarrow n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i} \overset{?}{=} 0$$

$$\Rightarrow n^2\cos^2 i \overset{?}{=} \frac{\mu^2}{\mu'^2}n'^2 - \frac{\mu^2}{\mu'^2}n^2\sin^2 i$$

显然无对应关系

3). $E_0'' = 0$ 的条件, 无反射?

$$n\cos i = \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}$$

$$n^2\cos^2 i + \frac{\mu^2}{\mu'^2}n^2(1-\cos^2 i) = \frac{\mu^2}{\mu'^2}n'^2$$

$$\cos^2 i\left(n^2 - \frac{\mu^2}{\mu'^2}n^2\right) = \frac{\mu^2}{\mu'^2}(n'^2 - n^2)$$

$$\Rightarrow \cos^2 i = \frac{\frac{\mu^2}{\mu'^2}(n'^2 - n^2)}{n^2 - \frac{\mu^2}{\mu'^2}n^2}$$

, $\mu = \mu'$, $\cos^2 i = 1$

$i = 0$ 或 π.

情形 (b).

① $\cos i\left(E_0 - E_0''\right) - \cos r\, E_0' = 0$ ①′

② 给出其余方程 (作业)

③ $\frac{1}{\mu}\left(k E_0 + k'' E_0''\right) - \frac{1}{\mu'} k' E_0' = 0$

$$\Longrightarrow \sqrt{\frac{\epsilon}{\mu}}\left(E_0 + E_0''\right) - \sqrt{\frac{\epsilon'}{\mu'}} E_0' = 0 \qquad ②'$$

④ 直接消去

$$\Longrightarrow E_0'' = E_0 - \frac{\cos r}{\cos i} E_0'$$

代入 ②′ 中

$$\sqrt{\frac{\epsilon}{\mu}}\left(2E_0 - \frac{\cos r}{\cos i} E_0'\right) - \sqrt{\frac{\epsilon'}{\mu'}} E_0' = 0$$

$$\Longrightarrow 2\sqrt{\frac{\epsilon}{\mu}} E_0 = \left(\sqrt{\frac{\epsilon}{\mu}}\frac{\cos r}{\cos i} + \sqrt{\frac{\epsilon'}{\mu'}}\right) E_0'$$

$$\Longrightarrow \frac{E_0'}{E_0} = \frac{2\sqrt{\frac{\epsilon}{\mu}}\cos i}{\sqrt{\frac{\epsilon}{\mu}}\cos r + \sqrt{\frac{\epsilon'}{\mu'}}\cos i}$$

$$= \frac{2nn'\cos i}{\frac{\mu}{\mu'} n'^2 \cos i + n\sqrt{n'^2 - n^2 \sin^2 i}}$$

$$\frac{E_0''}{E_0} = E_0 - \frac{2\sqrt{\frac{\epsilon}{\mu}}\cos r}{\sqrt{\epsilon}\cos r + \sqrt{\epsilon'}\cos i} E_0 = \frac{\sqrt{\frac{\epsilon'}{\mu'}}\cos i - \sqrt{\frac{\epsilon}{\mu}}\cos r}{\sqrt{\epsilon}\cos r + \sqrt{\epsilon'}\cos i} E_0 \quad ㊽$$

$$\frac{E_0''}{E_0} = \frac{\frac{\mu}{\mu'} n'^2 \cos i - n \sqrt{n'^2 - n^2 \sin^2 i}}{\frac{\mu}{\mu'} n'^2 \cos i + n \sqrt{n'^2 - n^2 \sin^2 i}}$$

分析：1). $E_0'' = 0$，无反射波

$$\frac{\mu}{\mu'} n'^2 \cos i = n \sqrt{n'^2 - n^2 \sin^2 i}$$

当 $\mu = \mu'$ 时，$n'^2 \cos i = n \sqrt{n'^2 - n^2 \sin^2 i}$

$$\Longrightarrow n'^4 \cos^2 i = n^2 \left(n'^2 - n^2 \sin^2 i \right)$$

$$\Longrightarrow n'^4 \cos^2 i + n^4 \left(1 - \cos^2 i\right) = n^2 n'^2$$

$$\Longrightarrow \left(n'^4 - n^4\right) \cos^2 i = n^2 \left(n'^2 - n^2 \right)$$

$$\Longrightarrow \cos^2 i = \frac{n^2}{n'^2 + n^2} = \frac{1}{(n'/n)^2 + 1}$$

当 $n'/n = 1.5$，$\cos i = 0.555$，$i = 56.3°$

这个角度被称为 Brewster 角

分析 why？

$$\frac{E_0'}{E_0} = \frac{2 n n' \cos i}{2 \frac{\mu}{\mu'} n'^2 \cos i} = \frac{\mu'}{\mu} \cdot \frac{n}{n'}$$

△

全部折射

2) Total internal reflection（全反射，无折射）

Snell's Law

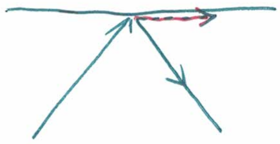
$$\frac{\sin i}{\sin r} = \frac{k'}{k} = \sqrt{\frac{\mu'\epsilon'}{\mu\epsilon}} = \frac{n'}{n}$$

临界角

若在$_{入}$ $i = i_0$, $r = \frac{\pi}{2}$, if $n > n'$

$$\sin i_0 = \frac{n'}{n} \Rightarrow i_0 = \sin^{-1}\left(\frac{n'}{n}\right)$$

$$\left\{\begin{array}{l} \dfrac{E_0'}{E_0} = \dfrac{2nn'}{\frac{\mu}{\mu'}n'^2} = \dfrac{2n}{\frac{\mu}{\mu'}n'} \\ \hline \dfrac{E_0''}{E_0} = 1 \end{array}\right.$$



What happens when $i > i_0$?

$$\sin r = \sin i \frac{n}{n'} > 1 , \ r = ?$$

$$\frac{\sin i}{\sin r} = \frac{k'}{k}$$, k' 为实 已经不可能 $\vec{k}' \to \cancel{ik'}\ \vec{k}'_{\parallel} \overset{+ik'_{\perp}}{\cancel{+ik'_{\perp}}}$

$\Longrightarrow k'\sin r = k'_{\parallel} = \sin i\, k$, 但是 $k'_{\perp}$ 要求<u>解</u>

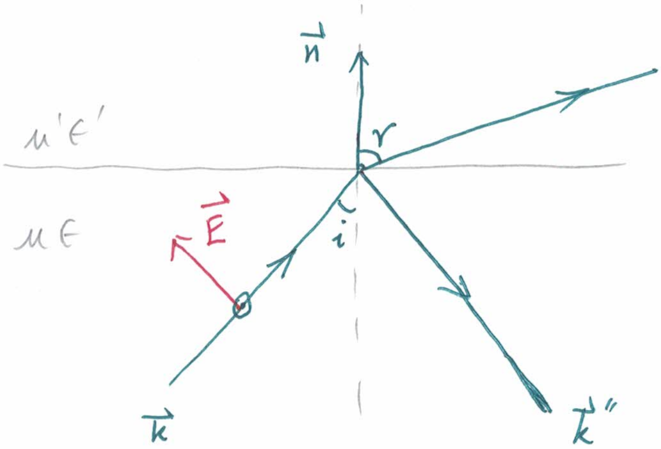
Snell's 定律对 $k'_{\perp}$ 无要求

~~替换: $n'\cos r \rightarrow i\sqrt{\left(\frac{\sin i}{\sin i_0}\right)^2-1}$~~

~~感受不到 n' 了~~

一些探究:

情形 (b)



$$\frac{E_0''}{E_0} \overset{\boxed{\mu=\mu'}}{=} \frac{n'^2\cos i - n\sqrt{n'^2-n^2\sin i}}{n'^2\cos i + n\sqrt{n'^2-n^2\sin^2 i}} = \boxed{\frac{\sqrt{\epsilon'}\cos i - \sqrt{\epsilon}\cos r}{\sqrt{\epsilon}\cos r + \sqrt{\epsilon'}\cos i}}$$

$$\frac{E_0'}{E_0} \overset{\boxed{\mu=\mu'}}{=} \frac{2nn'\cos i}{n'^2\cos i + n\sqrt{n'^2-n^2\sin i}} > 0.$$

$$\sqrt{\epsilon'}\cos i = \sqrt{\epsilon}\cos r$$

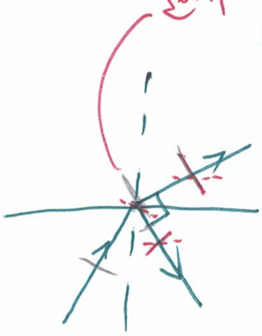
折射定律: $\frac{\sin i}{\sin r} = \sqrt{\frac{\epsilon'}{\epsilon}} = \frac{\cos r}{\cos i}$

$\Longrightarrow \quad \sin 2i = \sin 2r$

(1). $i_0 = r$.

(2) $i_0 = -r + \frac{\pi}{2} \Longrightarrow \boxed{i + r = \frac{\pi}{2}}$

边界上的振动方向?



$\vec{E}_{\parallel}$ 适

电场振荡方向垂直

当 $i < i_0$ 时, $E_0''/E_0 > 0$. $(n' > n)$

当 $i > i_0$ 时, E_0''/E_0 为负, 半波损失

$e^{i\vec{k}_{\parallel}'\cdot\vec{z} - k_\perp' z}$ evanescent waves.

情形 (b) $\vec{k}' = \vec{k}_{\parallel}' + i k_\perp' \vec{n}$

① $\cos i \, (E_0 - E_0'') - \cos\gamma \, E_0' = 0$

③ $\vec{k}' \times \vec{E}_0' = (\vec{k}_{\parallel}' + i k_\perp' \vec{z}) \times (\vec{E}_{0\parallel}' + E_{0z}\vec{z})$

$= \vec{k}_{\parallel}' \times \vec{z} \, E_{0z} + i k_\perp' \vec{z} \times \vec{E}_{0\parallel}'$

$\boxed{k_\perp = k\sqrt{\sin^2 i - \sin^2 i_0}}$ 怎样求出来？

$\sqrt{(ik_\perp)^2 + k_{\parallel}'^2} = k' = k\sin i_0$

↳ 总的波矢保持不变

$\Rightarrow -k_\perp^2 = k^2(\sin^2 i_0 - \sin^2 i)$

$\Rightarrow k_\perp = k\sqrt{\sin^2 i - \sin^2 i_0}$

why? evanescent wave.

$k = k'' = \omega\sqrt{\mu\epsilon}$

$k' = \sqrt{k_{\parallel}'^2 \mp k_\perp^2} = \omega\sqrt{\mu'\epsilon'} \underset{\uparrow}{=} \omega\sin i_0\sqrt{\mu\epsilon} = k\sin i_0$

而 $\sin i_0 = \sqrt{\frac{\mu'\epsilon'}{\mu\epsilon}} \Rightarrow \sqrt{\mu'\epsilon'} = \sin i_0\sqrt{\mu\epsilon}$ *

全反射：

$i = i_0$ 时，折射波沿界面传播.

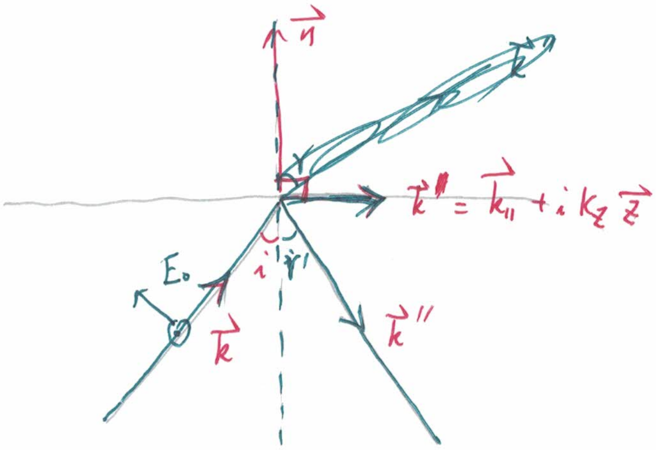
$$i_0 = \arcsin \frac{n'}{n}$$

$$\begin{cases} \frac{E_0'}{E_0} = \frac{2n}{\frac{\mu}{\mu'} n'} \\ \frac{E_0''}{E_0} = 1 \end{cases}$$

$i > i_0$，　$\vec{k} = \vec{k_{\parallel}} + i k_{\perp}$

$$k_{\perp} = k \sqrt{\sin^2 i - \sin^2 i_0}$$

$\vec{k_{\parallel}}$ 由动量守恒决定 $k \sin i$



折射波：$\vec{E}' = \vec{E_0'} e^{i\vec{k}' \cdot \vec{r} - i\omega t}$

$$= \vec{E_0'} e^{i\vec{k_{\parallel}'} \cdot \vec{r} - i\omega t} e^{-k_z z}$$

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

$$(\nabla\times\vec{E})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}E_k = \varepsilon_{ijk}\, i k_j E_k$$

$$\nabla\times\vec{E} = i\vec{k}\times\vec{E}$$

$$\vec{B}' = \sqrt{\mu'\epsilon'}\,\frac{\vec{k}'\times\vec{E}'}{k'}$$ 仍成立

情形(b) 两个边界条件

① $\cos i\,(E_0 - E_0'') = 0 \Longrightarrow E_0 = E_0''$

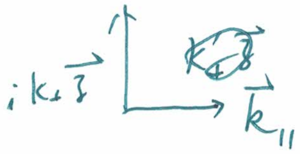
③ $\vec{H}_\parallel$ 连续

$$\left(\frac{1}{\mu}\sqrt{\mu\epsilon}\,\frac{\vec{k}\times\vec{E}_0}{k} + \frac{1}{\mu}\sqrt{\mu\epsilon}\,\frac{\vec{k}''\times\vec{E}_0''}{k} - \frac{1}{\mu'}\sqrt{\mu'\epsilon'}\,\frac{\vec{k}'\times\vec{E}_0'}{k'}\right)\times\vec{n} = 0$$

$$\Longrightarrow \frac{1}{\mu}\left(kE_0 + k''E_0''\right) - \frac{1}{\mu'}\underbrace{\vec{k}'\times\vec{E}_0'\times\vec{n}}_{\cancel{\neq k'E_0'}} = 0$$

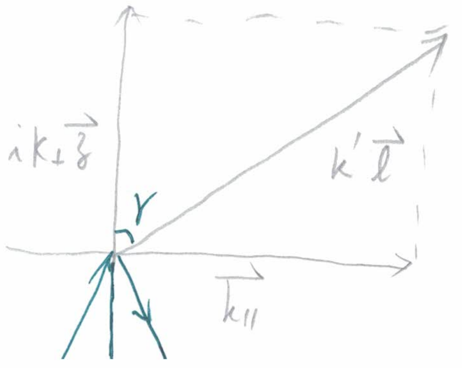
$$(\vec{k}'\times\vec{E}_0')_i = \varepsilon_{ijk}\,\vec{k}_j'\,\vec{E}_{0,k}' = \left(\vec{k}_\parallel' + i k_\perp'\vec{z}\right)\times\vec{E}_0'$$

复数：$\underbrace{\vec{k}\times\vec{E}}$



$\boxed{\text{复矢量}}$ 方向？

振幅？



$$ik_\perp\vec{z} + \vec{k}_\parallel = k'\vec{l}$$

$$\vec{l} = \frac{ik_\perp}{k'}\vec{z} + \frac{\vec{k}_\parallel}{k'}$$

这样可以定义 $\cos r$

$$\text{“}\cos r\text{”} = \frac{i k_1}{k'} = i \frac{k\sqrt{\sin^2 i - \sin^2 i_0}}{k'} = i\sqrt{\frac{\mu\epsilon}{\mu'\epsilon'}}\sqrt{\sin^2 i - \sin^2 i_0}$$

情形(b):
$$\frac{E_0'}{E_0} = \frac{2\sqrt{\frac{\epsilon}{\mu}}\cos i}{i\sqrt{\frac{\epsilon}{\mu}}\sqrt{\frac{\mu\epsilon}{\mu'\epsilon'}}\sqrt{\sin^2 i - \sin^2 i_0} + \sqrt{\frac{\epsilon'}{\mu'}}\cos i}$$

$$\underline{\underline{\mu=\mu'}} \quad \frac{2\sqrt{\epsilon}\cos i}{i\frac{\epsilon}{\sqrt{\epsilon'}}\sqrt{\sin^2 i - \sin^2 i_0} + \sqrt{\epsilon'}\cos i}$$

$$= \boxed{\frac{2\sqrt{\epsilon\epsilon'}\cos i}{i\epsilon\sqrt{\sin^2 i - \sin^2 i_0} + \epsilon'\cos i}}$$

$$\frac{E_0''}{E_0} = \frac{\sqrt{\frac{\epsilon'}{\mu'}}\cos i - \sqrt{\frac{\epsilon}{\mu}}\cos r \checkmark}{\sqrt{\frac{\epsilon}{\mu}}\cos r \checkmark + \sqrt{\frac{\epsilon'}{\mu'}}\cos i}$$ 复数 $\Longrightarrow$ 入射波
反射波
存在相位差

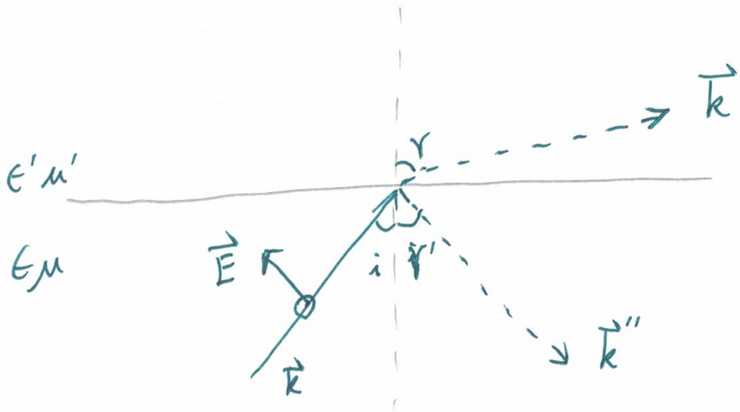
$$\underline{\underline{\mu=\mu'}} \quad \frac{\sqrt{\epsilon'}\cos i - \sqrt{\epsilon}\cos r}{\sqrt{\epsilon}\cos r + \sqrt{\epsilon'}\cos i}$$

$$= \frac{n'\cos i - n\cos r}{n\cos r + n'\cos i}$$

电磁波在导体表面的反射与折射

导体：介电常数 $\epsilon = \varepsilon' + i\varepsilon''$，$\varepsilon'' = \frac{\sigma}{\omega}$

从而在一定程度上改写了"反射、折射定律"



扩散方程
+ 边界条件

波动方程

动量守恒仍然满足，但是定律里面的 k 可能为复数。

$$\vec{k}_{\parallel} = \vec{k}'_{\parallel} = \vec{k}''_{\parallel}$$

$\vec{k}''$ 可以使得 $r' = i$，要求的是 r

在金属材料中，$k' = ?$

1) 金属中电磁波（准静态时）~~中~~遵从的是扩散方程。

~~$\frac{\partial}{\partial t}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} = \frac{1}{\mu\sigma_c}\nabla^2\begin{pmatrix}\vec{H}\\ L\end{pmatrix}$~~

$$\frac{\partial}{\partial t}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} = \frac{1}{\mu\sigma_c}\nabla^2\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix}$$

不是波动方程。

$\Downarrow$ $\epsilon \approx i\varepsilon''$，纯虚数。

2) 边界条件是什么？

$$\begin{cases}\vec{n}\times(\vec{E}_2 - \vec{E}_1) = 0\\ \vec{n}\times(\vec{H}_2 - \vec{H}_1) = \vec{\alpha}_f\\ \vec{n}\cdot(\vec{D}_2 - \vec{D}_1) = \sigma_f\end{cases}$$

准静态时

$\sigma_f = 0$，$E_n = 0$

$J_n = 0$

导体内 $\vec{E}' = \vec{E}_0' e^{i\vec{k}_{\parallel}'\cdot\vec{\rho} + ik_z' z - i\omega t}$

$$\to -i\omega\vec{E}_0' = \frac{1}{\mu\sigma_c}\left(-k_x'^2 - k_y'^2 - k_z'^2\right)\vec{E}_0'$$

$$\to \omega = -i\left(k_x'^2 + k_y'^2 + k_z'^2\right)\frac{1}{\mu\sigma_c} = -ik'^2\frac{1}{\mu\sigma_c}$$

$$\boxed{\omega = \frac{k'}{\sqrt{\mu'\epsilon'}}}$$

$$\epsilon' = i\frac{\sigma_c}{\omega}$$

$$\to \omega^2 = -i\frac{k'^2}{\mu'\sigma_c/\omega} \to \omega^3 = -i\frac{k'^2}{\mu'\sigma_c}$$

讨论 paradox:

$$\begin{cases} \nabla\cdot\vec{E} = 0 \\ \nabla\times\vec{E} = i\omega\mu\vec{H}(\omega) \\ \nabla\cdot\vec{H} = 0 \\ \nabla\times\vec{H} = -i\omega\epsilon(\omega)\vec{E}(\omega) \end{cases}$$

复数

含 ω

~~问题出在哪里?~~

~~$\vec{D}(t) = \int \epsilon(t-t')\vec{E}(t')$~~

$$\Longrightarrow \nabla\times\nabla\times\vec{E} = i\omega\mu\,\nabla\times\vec{H} = \omega^2\mu\epsilon(\omega)\vec{E}(\omega)$$

$$\downarrow$$

$$\Longrightarrow -\nabla^2\vec{E} = k^2\vec{E} = \omega^2\mu\left(\varepsilon' + i\frac{\sigma}{\omega}\right)\vec{E}(\omega)$$

$$\Longrightarrow k^2 = \mu\omega^2\varepsilon' + i\mu\sigma\omega$$; 一元二次方程

$\omega = -ik^2/(\sigma\mu)$

运用扩散方程

$$\boxed{i\omega\mu\sigma_c - k_{\parallel}'^{2} = k_z'^{2}} \Longrightarrow \underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim}{k_z' = \sqrt{i\omega\mu\sigma_c - k_{\parallel}'^{2}}} = \sqrt{i\omega\mu\sigma_c - k_{\parallel}^{2}}$$

求振幅 $\vec{E}_0'$

将 $\cos\vec{\gamma}$ 给求出来？

$$\text{“}\cos\gamma\text{”} = \frac{k_z}{k'} = \frac{k_z}{\sqrt{i\omega\mu\sigma_c}} = \frac{\sqrt{i\omega\mu\sigma_c - k_{\parallel}^{2}}}{\sqrt{i\omega\mu\sigma_c}}$$

$k_{\parallel}$ 与 $i\omega\mu\sigma_c$ 比较

~~ω_p ω_0~~ $\qquad \omega = ck_{\parallel}$

$k_{\parallel}$ 与 $c\mu\sigma_c$ 比较：

$\downarrow$

$\gg 1$

$c = 3\times10^{8}\ \mathrm{m/s}$

$\mu = 4\pi\times10^{-7}$

$\sigma_c = 5.9\times10^{7}\ \Omega^{-1}\cdot\mathrm{m}^{-1}$

“$\cos\gamma$” ≈ 1

前面的结果“可以使用”

情形 (b)：$\mu = \mu'$

$$\frac{E_0'}{E_0} = \frac{2\sqrt{\epsilon}\cos\theta_i}{\sqrt{\epsilon}\cos\gamma + \sqrt{\epsilon'}\cos\theta_i} = \frac{2\sqrt{\epsilon}\ \underset{\sim\sim}{\cos}\theta_i}{\sqrt{\epsilon} + \sqrt{i\sigma_c/\omega}\cos\theta_i}$$

垂直入射：$i = 0$，$\sqrt{i} = \sqrt{e^{i\frac{\pi}{2}}} = e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$$\left.\frac{E_0'}{E_0}\right|_{i} = \frac{2\sqrt{\epsilon}}{\sqrt{\epsilon} + \sqrt{\frac{\sigma_c}{\omega}}\,\frac{\sqrt{2}}{2}(\ \ )} \approx \sqrt{2}\sqrt{\frac{\epsilon\omega}{\sigma}}\,(1-i)$$

$\frac{\epsilon\omega}{\sigma} \ll 1$，$E_0'$ 非常小，大部分光将反射

透射系数 $|T|^2 = \left|\frac{E_0'}{E_0}\right|^2 = 4\frac{\epsilon\omega}{\sigma_c} \ll 1$

$$\frac{E_0''}{E_0} \overset{\mu=\mu'}{=} \frac{\sqrt{\epsilon'}\cos i - \sqrt{\epsilon}\cos r}{\sqrt{\epsilon}\cos r + \sqrt{\epsilon'}\cos i}$$

$$= \frac{\sqrt{\epsilon'}\cos i - \sqrt{\epsilon}}{\sqrt{\epsilon'}\cos i + \sqrt{\epsilon}}$$

$$\sqrt{\epsilon'} = \sqrt{\frac{\sigma_c}{\omega}}\frac{\sqrt{2}}{2}(1+i)$$

$$i=0 \quad = \frac{\sqrt{\epsilon'} - \cancel{1}\sqrt{\epsilon}}{\sqrt{\epsilon'} + \cancel{0}\sqrt{\epsilon}}$$

$$= \frac{\sqrt{\frac{\sigma_c}{2\omega}}\cancel{\frac{\sqrt{2}}{2}}(1+i) - \cancel{1}\sqrt{\epsilon}}{\sqrt{\frac{\sigma_c}{2\omega}}\cancel{\frac{\sqrt{2}}{2}}(1+i) + \cancel{1}\sqrt{\epsilon}}$$

$$= -\frac{1+i - \sqrt{\frac{2\omega\epsilon}{\sigma_c}}}{1+i + \sqrt{\frac{2\omega\epsilon}{\sigma_c}}} \nearrow \frac{\cancel{1} - \sqrt{\frac{2\omega\epsilon}{\sigma_c}} + \frac{\omega\epsilon}{\sigma_c}}{1 + \sqrt{\frac{2\omega\epsilon}{\sigma_c}} + \frac{\omega\epsilon}{\sigma_c}}$$

$$= \left(1 - \sqrt{\frac{2\omega\epsilon}{\sigma_c}} + \frac{\omega\epsilon}{\sigma_c}\right)\left(1 - \sqrt{\frac{2\omega\epsilon}{\sigma_c}} - \frac{\omega}{\sigma_c}\right.$$

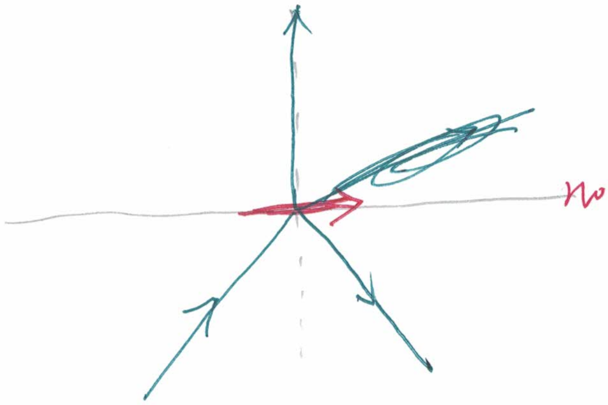
反射系数

$$R = \left|\frac{E_0''}{E_0}\right|^2 \approx 1 ~~- 2\sqrt{\frac{2\omega\epsilon}{\sigma_c}}~~ \to \frac{\left(1 - \sqrt{\frac{2\omega\epsilon}{\sigma_c}}\right)^2 + 1}{\left(1 + \sqrt{\frac{2\omega\epsilon}{\sigma_c}}\right)^2 + 1} \approx 1 - 2\sqrt{\frac{2\omega\sigma_c}{\sigma}}\cancel{\theta} + \frac{2\omega}{\sigma_c}$$

$$T + R ~~\ne 1~~ = 1 - 2\sqrt{\frac{2\omega\epsilon}{\sigma_c}} + \frac{\cancel{6}\epsilon\omega}{\sigma_c} ~~= \left(1 - \sqrt{\frac{2\omega\epsilon}{\sigma_c}}\right)^2 + \frac{2\epsilon\omega}{\sigma_c}~~$$

$$\ne 1$$

边界条件怎么满足?



近似 no normal component!

↓

$$``\cos\gamma" = \frac{\sqrt{i\omega\mu\sigma_c - k_{\parallel}^2}}{\sqrt{i\omega\mu\sigma_c}}$$

怎样更好满足边界条件

表面电阻:

面电流! (耗散能量)

$$\vec{E}' = E_0' \vec{x}\, e^{i\vec{k}_{\parallel}\cdot\vec{g} - k_z z - i\omega t}$$

$$\simeq \sqrt{\frac{2\epsilon\omega}{\sigma_c}}(1-i)\,\vec{x}\, e^{i\vec{k}_{\parallel}\cdot\vec{g} - k_z z - i\omega t}$$

$$\vec{J} = \sigma_c \vec{E}'$$

$$\vec{\alpha}_f = \int_0^\infty \vec{J}\, dz = \sqrt{2\epsilon\omega\sigma_c}(1-i)\,\vec{x}\,\frac{1}{k_z}$$

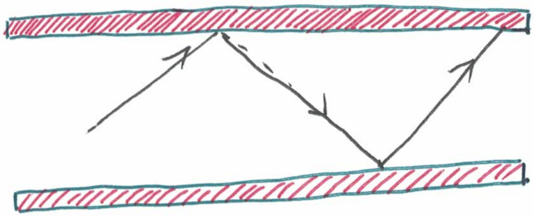
$$= \frac{\sqrt{2\epsilon\omega\sigma_c}(1-i)}{\sqrt{i\omega\mu\sigma_c}}\vec{x} = \frac{1-i}{\frac{\sqrt{2}}{2}(1+i)}\sqrt{\frac{2\epsilon}{\mu}}\,\vec{x}$$

$$= (2-2i)\sqrt{\frac{\epsilon}{\mu}}\,\vec{x}$$

面电流 损耗能量. ~~式 $\int E^* E\, dx$~~

波导与谐振腔

电介质，金属都可以全反射电磁波



金属或电介质 ⟹ 用来低耗散传输电磁波 ⟹ 波导

并不是所有频率、波矢的波均能传播

⟹ $\boxed{\text{模式问题}}$

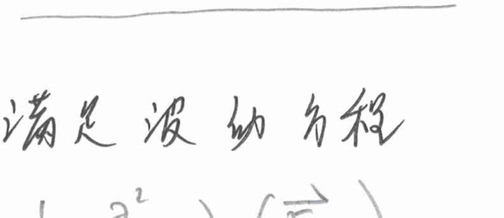
将金属看成完美导体 $\sigma \to \infty$，趋肤深度为0，则导体内无电场

$\boxed{\vec{n} \cdot \vec{D} = \Sigma}$，$n \times \vec{H} = \vec{k}$

边界条件：在导体边界上 $\boxed{E_{\parallel} = 0}$，$\nabla \cdot \vec{E} = 0 \Longrightarrow \left.\frac{\partial E_n}{\partial n}\right|_s = 0$

$B_{\perp} = 0 \Longrightarrow \boxed{H_{\perp}|_s = 0}$

$\vec{H} = -\frac{i}{\omega\mu} \nabla \times \vec{E}$?



$\vec{H}$ 在边界附近只能平行于壁，而 $\vec{E}$ 只能垂直于壁！

重要的图像

波导内，腔内满足波动方程

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\vec{E} \\ \vec{B}\end{pmatrix} = 0$$

~~$\nabla \times \vec{E} = i\omega B$~~

$\nabla \cdot \vec{E} = 0$, $\nabla \cdot \vec{B} = 0$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix}$$

$$= \cancel{\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_z}{\partial x}\right)\vec{k}}$$

$= \partial E_x \, \vec{i} - \partial E_x \, \vec{k}$

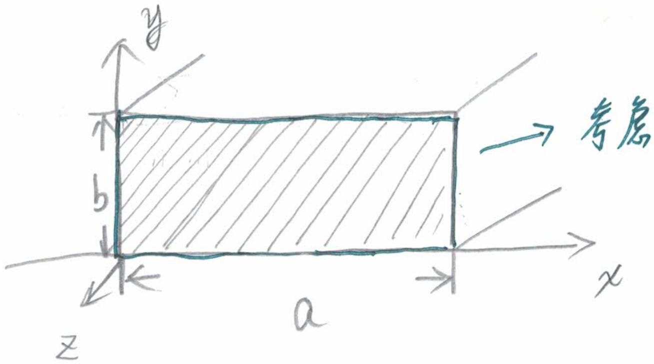
波导沿 $\vec{z}$ 方向平移不变

$$\vec{E}(x,y,z) = \vec{E}(x,y)\, e^{ikz - i\omega t}$$

$$\frac{\partial^2}{\partial x^2}\vec{E} + \frac{\partial^2}{\partial y^2}\vec{E} - k^2\vec{E} + \frac{\omega^2}{c^2}\vec{E} = 0$$

$$\frac{\omega}{c} = k_c \Longrightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\vec{E} + (k_c^2 - k^2)\vec{E} = 0$$

x, y 方向可用傅利叶级数展开



→ 考虑一个矩形波导.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_i(x,y) + (k_c^2 - k^2)\, E_i(x,y) = 0$$

$$E_i(x,y) = X(x)\, Y(y)$$

$$\frac{\partial^2 X(x)}{\partial x^2} Y(y) + \frac{\partial^2 Y(y)}{\partial y^2} X(x) + (k_c^2 - k^2)\, X(x)\, Y(y) = 0$$

$$\Longrightarrow \underset{\substack{||| \\ -k_x^2}}{\frac{\partial^2 X(x)}{\partial x}\frac{1}{X(x)}} + \underset{\substack{||| \\ -k_y^2}}{\frac{\partial^2 Y(y)}{\partial y^2}\frac{1}{Y(y)}} + k_c^2 - k^2 = 0$$

$$\Longrightarrow k_x^2 + k_y^2 + k^2 = k_c^2, \quad \frac{\partial^2 X(x)}{\partial x^2} + k_x^2 X(x) = 0, \quad \frac{\partial^2 Y(y)}{\partial y^2} + k_y^2 Y(y) =$$

$$X(x) = A_1 \cos(k_x x) + B_1 \sin(k_x x)$$

$$Y(y) = A_2 \cos(k_y y) + B_2 \sin(k_y y)$$

$$\Longrightarrow E_i(x,y) = \Big(A_{1i} \cos(k_x x) + B_{1i} \sin(k_x x)\Big) \times \Big(A_{2i} \cos(k_y y) + B_{2i} \sin(k_y y)\Big)$$

再根据边界条件定义 $\{A_{1i}, B_{1i}\}$ 关系，$\{A_{2i}, B_{2i}\}$ 关系 k_x，k_y 可取的值等

$\boxed{\text{求 } E_x(x,y)}$

$$\left.\begin{array}{l} E_x(x, y=0) \underset{\textcircled{1}}{=} 0, \quad E_x(x, y=b) \underset{\textcircled{2}}{=} 0 \\ \dfrac{\partial E_x(x=0, y)}{\partial x} \underset{\textcircled{3}}{=} 0, \quad \dfrac{\partial E_x(x=a, y)}{\partial x} \underset{\textcircled{4}}{=} 0 \end{array}\right\}$$

4个边界条件

这个本质上不是边界条件

① $(A_{1x} \cos(k_x x) + B_{1x} \sin(k_x x)) A_{2x} = 0 \Longrightarrow A_{2x} = 0$

② $(A_{1x} \cos(k_x x) + B_{1x} \sin(k_x x)) B_{2x} \sin(k_y b) = 0 \Longrightarrow k_y = \dfrac{n\pi}{b}, \ n = 0, 1, \dots$

③ $k_x B_{1x} (A_{2x} \cos(k_y y) + B_{2x} \sin(k_y y)) = 0 \Longrightarrow B_{1x} = 0$

④ $k_x A_{1x} \sin(k_x a) (A_{2x} \cos(k_y y) + B_{2x} \sin(k_y y)) = 0 \Longrightarrow k_x = \dfrac{m\pi}{a}$

$m = 0, 1, \dots$

$$\Longrightarrow \boxed{\begin{aligned} E_x(x,y) &= C_x \cos(k_x x) \sin(k_y y) \\ E_y(x,y) &= C_y \sin(k_x x) \cos(k_y y) \\ E_z(x,y) &= C_z \sin(k_x x) \sin(k_y y) \end{aligned}}, \quad \left(k_x = \frac{m\pi}{a}, \ k_y = \frac{n\pi}{b}\right)$$

C_x, C_y, C_z 并不独立.

$$\nabla\cdot\vec{E}=0 \Longrightarrow C_1 k_x + C_2 k_y - i C_z k = 0$$

同样可以求出磁场模式

$$\begin{cases} H_x(x,y) = C_x' \sin(k_x x)\cos(k_y y) \\ H_y(x,y) = C_y' \cos(k_x x)\sin(k_y y) \\ H_z(x,y) = C_z' \cos(k_x x)\cos(k_y y) \end{cases}$$

模式：(1) 若 $C_z = 0$，则 $E_z = 0$

$$\Longrightarrow C_1 k_x + C_2 k_y = 0 \Rightarrow -\frac{k_x}{k_y} = \frac{C_1}{C_2}$$

$$\Longrightarrow H_z \neq 0. \quad (因\ \vec{H} = -\frac{i}{\omega\mu}\nabla\times\vec{E})$$

TE 模

(1). $C_z' = 0$，则 $H_z = 0$，$\boxed{\nabla\cdot\vec{H}=0}$

TM 模

The modes are divided into two classes.

(1). transverse magnetic modes $H_z = 0$, $E_z|_s = 0$

$\Longrightarrow \{E_x, E_y, H_z\}$

(2). transverse electric field $E_z = 0$, $\left.\frac{\partial B_z}{\partial n}\right|_s = 0$

$\Longrightarrow \{H_x, H_y, E_z\}$

为什么总是分成两个 category?

$$E_z = 0 \longrightarrow H_z \neq 0 \quad \text{TM 模}$$

$$\underline{E_z \neq 0} \Rightarrow \text{TE 模} \ (H_z = 0,\ H_z \neq 0)$$

D 是否存在 $E_z \neq 0$, $H_z \neq 0$ 这种情况

色散关系： $\omega \sim k$

$$k_x^2 + k_y^2 + k^2 = k_c^2$$

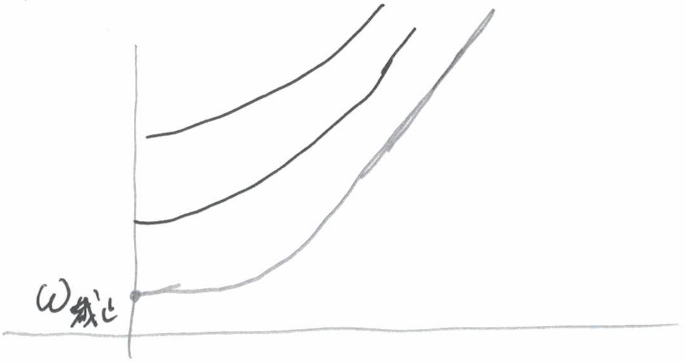
$$\Longrightarrow \omega = c\sqrt{k_x^2 + k_y^2 + k^2}$$

$$= c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k^2}$$

m, n 是否可以同时为0.

不可能，因为此时 $E_x(x,y)$, $E_z(x,y)$ 均为0.

此时，$\boxed{\omega_{k=0}}$ 存在一个最小值，称为截止频率



对于 a, b ~ 10cm

$\omega \sim$ GHz

当 a>b 时，m=1, n=0 的频率最低，相应的模式称为基模.

此时 $E_x = 0$, $E_z = 0$. 电场只有 y 分量.

$$E_y = C_y \sin\left(\frac{\pi x}{a}\right) e^{ikz}$$

$$\vec{H} = -\frac{i}{\omega\mu}\nabla\times\vec{E}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

~~$H_\alpha = \epsilon_{\alpha\beta} \frac{\partial}{\partial x_\beta} E_y$~~

~~$H_x = \left(-\frac{i}{\omega\mu}\right)\frac{\partial}{\partial z} E_y$~~

$H_y = 0$

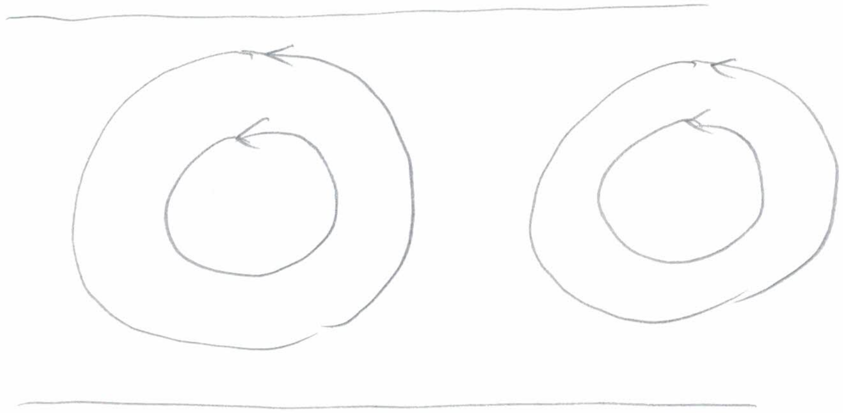
$H_z = 0$

$$H_\alpha = -\frac{i}{\omega\mu}\,\varepsilon_{\alpha\beta y}\,\frac{\partial}{\partial x_\beta}E_y$$

$$H_y = 0$$

$$H_x = \frac{i}{\omega\mu}\frac{\partial}{\partial z}E_y = \overset{\nearrow}{+}\,\frac{k}{\omega\mu}C_y\sin\left(\frac{\pi x}{a}\right)e^{ikz}$$

$$H_z = -\frac{i}{\omega\mu}\frac{\partial}{\partial x}E_y = -\frac{i}{\omega\mu}\frac{\pi}{a}C_y\cos\left(\frac{\pi x}{a}\right)e^{ikz}$$



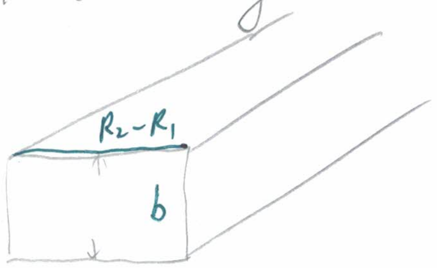
polarization-momentum locking

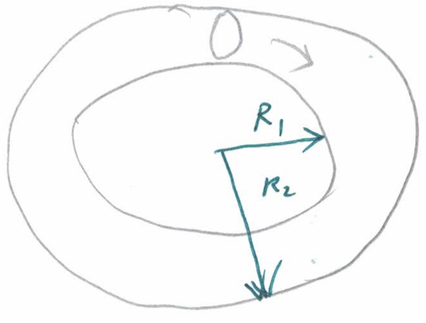
磁场具有涡旋

谐振腔

将另外两个方向围起来形成谐振腔

Torus cavity





要有目的性地去求场的性质，$\boxed{\text{TM 模}\ (E_z \neq 0)}$ → 作业：求一下 TE 模（提示从 $H_z \neq 0$ 开始）

$H_z = 0$

66

$$\vec{E} = \vec{E}_t + \vec{E}_z \Rightarrow \vec{E}_z = \psi(\rho,\phi)\cos\left(\frac{p\pi z}{b}\right),\quad p=0,1,2,\ldots$$

$\vec{E}_t$ 可以存在（~~不需求~~ 可由 E_z 求出）

模式的进一步分析:

$$\nabla\times\vec{E}=i\omega\vec{B}\,,\qquad \nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{B}=-i\mu\epsilon\omega\vec{B}\,,\quad \nabla\cdot\vec{E}=0$$

$$\vec{E}=E_z\vec{z}+\vec{E}_t\,,\quad \vec{B}=B_z\vec{z}+\vec{B}_t$$

$$\Longrightarrow\left\{\begin{array}{l}\dfrac{\partial\vec{E}_t}{\partial z}+i\omega\hat{z}\times\vec{B}_t=\nabla_t E_z\,,\quad \vec{z}\cdot(\nabla_t\times\vec{E}_t)=i\omega B_z\\[2ex] \dfrac{\partial\vec{B}_t}{\partial z}-i\mu\varepsilon\omega\hat{z}\times\vec{E}_t=\nabla_t B_z\,,\quad \vec{z}\cdot(\nabla_t\times\vec{B}_t)=-i\omega\mu\epsilon E_z\\[2ex] \nabla_t\cdot\vec{E}_t=-\dfrac{\partial E_z}{\partial z}\,,\quad \nabla_t\cdot\vec{B}_t=-\dfrac{\partial B_z}{\partial z}\end{array}\right.$$

$\boxed{E_z\,,B_z}$ 知道了，$\vec{E}_t$, $\vec{B}_t$ 就知道了

$E_z=0$, $B_z\neq0$, TE

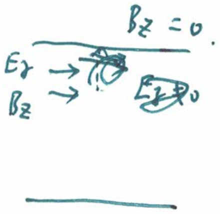
$E_z\neq0$, $B_z=0$, TM

$E_z=0$, $B_z=0$ $\Longrightarrow$ 真正的TEM $\Longrightarrow$ $\vec{B}_t,\vec{E}_t=0$.

不行

$\underbrace{E_z\neq0\,,\quad B_z\neq0\,,}$ 为什么不行

边界条件: $\vec{E}_{\parallel}$ 连续, $\vec{B}_{\perp}$ 连续



复习: $\begin{cases} \text{贝塞尔方程} \\ \text{虚宗量贝塞尔方程} \\ \text{球贝塞尔方程.} \end{cases}$

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + \begin{cases} (x^2-m^2) \\ (-x^2-m^2) \\ (k^2x^2-l(l+1)) \end{cases} * R = $$

通解: $y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x)$

$y(x) = C_1 J_\nu(x) + C_2 N_\nu(x)$

满足方程

$$(\nabla_t^2 + \gamma_p^2)\psi_p = 0$$

$$\gamma_p^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{p\pi}{b}\right)^2$$

在柱坐标系下:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \gamma_p^2\psi = 0$$

将 $\psi(\rho,\phi) = R_m(\rho)e^{im\phi}$, $m\in\mathbb{Z}$.

$\hookrightarrow$ 周期性边界条件, m取整.

得到 Bessel 方程 (m阶)

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R_m(\rho)}{\partial\rho}\right) + \left(\gamma_{p,m}^2 - \frac{m^2}{\rho^2}\right)R_m(\rho) = 0$$

$$\Rightarrow R_m(\rho) = \underbrace{J_m(\gamma_{p,m}\rho)}_{\text{第一类 Bessel 函数}} + C\underbrace{N_m(\gamma_{p,m}\rho)}_{\text{第二类}}$$

R_1, R_2 处电场为0, 作为边界条件. (不允许电场存在)

$$J_m(\gamma_{p,m}R_2) + CN_m(\gamma_{p,m}R_2) = 0$$

$$J_m(\gamma_{p,m}R_1) + CN_m(\gamma_{p,m}R_1) = 0$$

$$\Rightarrow C = -\frac{J_m(\gamma_{p,m}R_1)}{N_m(\gamma_{p,m}R_1)}$$

fsolve

和本征值方程: $J_m(\gamma_{p,m}R_2)N_2(\gamma_{p,m}R_1) = J_m(\gamma_{p,m}R_1)N_m(\gamma_{p,m}$

$\omega_m = c\gamma_{p=0,m} = \{10.84, 11.87, 13.16\}$ GHz

when $R_1 = 15mm$, $R_2 = 30mm$

matlab

磁场: $\vec{H}(\rho,\phi) = H_\rho \vec{e_\rho} + H_\phi \vec{e_\phi}$

$$H_\rho(\rho,\phi,z) = \frac{1}{\mu_0\gamma_m c}\frac{m}{\rho}\mathcal{E}_z$$

$$H_\phi(\rho,\phi) = -i\frac{1}{\mu_0\gamma_m c}\frac{\partial \mathcal{E}_z}{\partial \rho}$$

作业：计算矩形腔{a, b, c}的各种模式

$R_1 \to 0$ 时 成为无洞的 cavity，解如何变化.

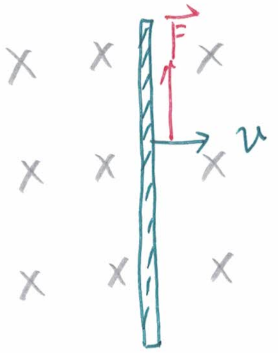
$x \to 0$ 时，

$$\boxed{\begin{aligned} &J_0(x) \to 1 \\ &J_\nu(x) \to 0 \\ &J_{-\nu}(x) \to \infty \\ &N_0(x) \to -\infty \\ &N_\nu(x) \to \pm\infty \end{aligned}}$$

$(\nu \neq 0)$

抄数学~~数~~物理方法书

动体的电动力学

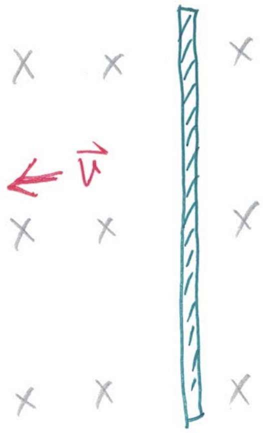


$$\vec F = q\vec v \times \vec B = q\vec E'$$

$$\vec E' = \vec v \times \vec B \quad (等效电场)$$

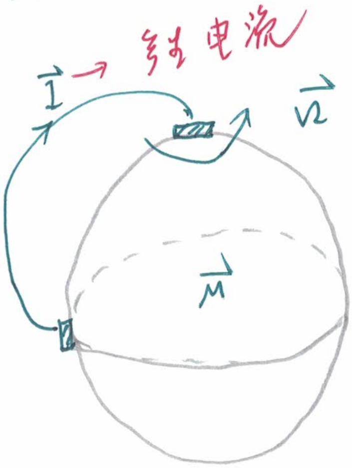
似乎从导线的坐标系看，电子感受到的磁场力 $\vec E = \vec E_0 + \vec E'$

相对于场运动



what is happening ?

单极感应现象



伽利略变换？ 怎么进行？

$(t, x, y, z) \Longleftrightarrow (t', x', y', z')$

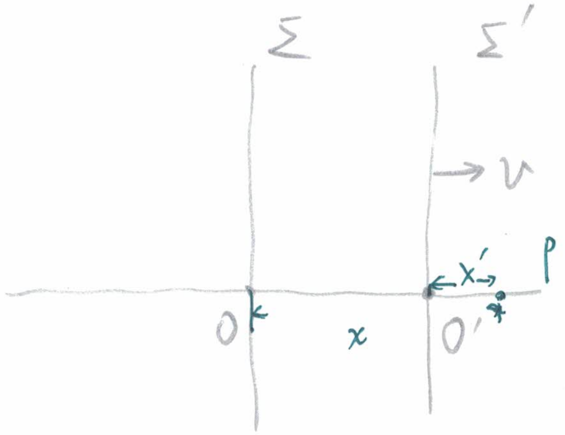
惯性参考系之间的变换

t=0 两坐标系重合

K' 相对于 K 以 $\vec{v}$ 运动

$$t' = t$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} v_x t \\ v_y t \\ v_z t \end{pmatrix}$$



$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline -v_x & 1 & & \\ -v_y & & 1 & \\ -v_z & & & 1 \end{array}\right) \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

⇓

变换矩阵

其它物理量，如速度、角动量等变换规律如何得到！

标量	~~场~~	φ
矢量	~~场~~	$\vec{A}$
张量	~~场~~	$\vec{T}$

$$\frac{\partial x'_\mu}{\partial x_\nu}$$

$$T_{\mu\nu} = \left(\begin{array}{c|ccc} \frac{\partial t'}{\partial t} & \frac{\partial t'}{\partial x} & & \\ \hline \frac{\partial x'}{\partial t} & \frac{\partial x'}{\partial x} & & \\ \frac{\partial y'}{\partial t} & & & \\ \frac{\partial z'}{\partial t} & & & \end{array}\right)$$

标量类 $\varphi' = \varphi$ ，不变

矢量类：$A'_{\mu} = \Gamma_{\mu\nu} A_{\nu}$

即
$$\begin{pmatrix} 1 \\ A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \frac{\partial t'}{\partial t} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} 1 \\ A_x \\ A_y \\ A_z \end{pmatrix}$$

如速度
$$\begin{pmatrix} 1 \\ v'_x \\ v'_y \\ v'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -V_x & 1 & & \\ -V_y & & 1 & \\ -V_z & & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} 1 \\ V_x - V_x \\ V_y - V_y \\ V_z - V_z \end{pmatrix}$$

不涉及时间的矢量，（如力）按照规则
$$\begin{pmatrix} 0 \\ F'_x \\ F'_y \\ F'_z \end{pmatrix} = \Gamma_{\mu\nu} \begin{pmatrix} 0 \\ F_x \\ F_y \\ F_z \end{pmatrix}$$ 变换

张量类：$\overleftrightarrow{T} = T_{ij}$ ，按照 $T'_{\alpha\beta} = \Gamma_{\alpha\mu} T_{\mu\nu} \Gamma_{\nu\beta}$ 变换

可见知道了变换规则，进行变换变得容易

场：$\begin{cases} 标量场\ \varphi(x,y,z,t) \\ 矢量场\ \vec{A}(x,y,z,t) \\ 张量场\ \overleftrightarrow{T}(x,y,z,t) \end{cases}$ ，依赖于时空 $\boxed{(x,y,z,t)}$

$\Downarrow$

也要变换

$$\varphi'(\overset{x'_\mu}{\cancel{x',y',z',t'}}) = \varphi(T_{\mu\nu}\, x_\nu)$$

$$A'_\alpha(x'_\mu) = T_{\alpha\beta}\, A_\beta(T_{\mu\nu}\, x_\nu)$$

等等.

$\downarrow \begin{pmatrix} E'_x(x',y',z') \\ E'_y(x',y',z') \\ E'_z(x',y',z') \end{pmatrix}$ 如何变换

$\boxed{考虑一个1维情况就可以了}$

$$\begin{pmatrix} 0 \\ E'_x(x') \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -v_x & 1 \end{pmatrix} \begin{pmatrix} 0 \\ E_x(x-v_x t) \end{pmatrix}$$

$$E'_x(x') = E_x(x-v_x t)$$

仅仅是运动的场运动过来

对于均匀的场，$E'_x(x') = E_x(x)$，不变！

可见伽利略变换与电动力学有一定矛盾！

基本方程的变换:

例如标量场 $\psi(x,y,z,t)$ 所满足的波动方程,

在K惯性系下,满足波动方程

$$\frac{\partial^2\psi}{\partial t^2} - v^2\nabla^2\psi = 0$$

在K'惯性系下,满足什么方程

$$\begin{cases} \psi(x'_\mu) = \psi(\Gamma_{\mu\nu}x_\nu) \\ \frac{\partial}{\partial x'_\mu} = \frac{\partial}{\partial x_\nu}\frac{\partial x_\nu}{\partial x'_\mu} = \frac{\partial x_\nu}{\partial x'_\mu}\frac{\partial}{\partial x_\nu} = (\Gamma^{-1})^{\nu}_{\ \mu}\frac{\partial}{\partial x_\nu} \end{cases}$$

$$\psi(x_\nu) = \psi'(\Gamma^{-1}_{\nu\mu}x'_\mu)$$

$$\frac{\partial}{\partial x_\nu} = \Gamma_{\nu\mu}\frac{\partial}{\partial x'_\mu}$$

$$\frac{\partial^2}{\partial t'^2}\psi'(\Gamma^{-1}_{\mu\nu}x'_\mu) - v^2\,\Gamma_{\nu\mu}\frac{\partial}{\partial x'_\mu}\Gamma_{\nu\delta}\frac{\partial}{\partial x'_\delta}\psi'(\Gamma^{-1}_{\mu\nu}x'_\mu) = 0$$

$$\Longrightarrow \frac{\partial^2}{\partial t'^2}\psi'(\Gamma^{-1}_{\mu\nu}x'_\mu) - \underbrace{\Gamma_{\nu\mu}\Gamma_{\nu\delta}}\,v^2\frac{\partial}{\partial x'_\mu}\frac{\partial}{\partial x'_\delta}\psi'(\Gamma^{-1}_{\mu\nu}x'_\mu) = 0$$

不妨讨论,一维最简单

$$\frac{\partial}{\partial t'^2}\psi'(t', \overset{V_x t'+x'}{\cancel{V_x x}}) - \underbrace{\Gamma_{xx}\Gamma_{xx}}\,v^2\frac{\partial^2}{\partial x'^2}\psi'(t', \overset{V_x t'+x'}{\cancel{V_x x'}}) = 0$$

$$\rightarrow \frac{\partial}{\partial t'^2}\psi'(t', V_x t'+x') - v^2\frac{\partial^2}{\partial x'^2}\psi'(t', V_x t'+x') = 0$$

$$\frac{\partial}{\partial t'} \psi'(t', V_x t' + x')$$

$$= \frac{\partial}{\partial t'} \psi'(t', x') + V_x \frac{\partial}{\partial x'} \psi'(t', x')$$

$$\frac{\partial^2}{\partial t'^2} \psi'(t', V_x t' + x')$$

$$= \frac{\partial^2}{\partial t'^2} \psi'(t', x') + 2V_x \frac{\partial}{\partial t'} \frac{\partial}{\partial x'} \psi'(t', x') + V_x^2 \frac{\partial}{\partial x'} \psi'(t', x')$$

$\Longrightarrow$ 波动方程

$$\underbrace{\left(\frac{\partial}{\partial t'} + V_x \frac{\partial}{\partial x'}\right)}^{2} \psi'(t', x') - v^2 \frac{\partial^2}{\partial x'^2} \psi'(t', x') = 0$$

$$\begin{pmatrix} \frac{\partial}{\partial t'} \\ \frac{\partial}{\partial x'} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -V_x & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \end{pmatrix}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t'} + V_x \frac{\partial}{\partial x'}$$

复习:

坐标变换 $(t, x, y, z) \Longleftrightarrow (t', x', y', z')$

$$x'_\mu = \Gamma_{\mu\nu}\, x_\nu$$

$$\Gamma_{\mu\nu} = \left(\begin{array}{c|ccc} \frac{\partial t'}{\partial t} & \frac{\partial t'}{\partial x} & \frac{\partial t'}{\partial y} & \frac{\partial t'}{\partial z} \\ \hline \frac{\partial x'}{\partial t} & \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial t} & \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial t} & \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{array}\right)$$

伽利略变换，该矩阵为

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline -V_x & 1 & 0 & 0 \\ -V_y & 0 & 1 & 0 \\ -V_z & 0 & 0 & 1 \end{array}\right)$$

伽利略变换的时空特点

(1). 空间距离不变

(2). 时间距离不变

(3). 时间顺序不变

是逆还是不是逆

场的变换 $A'_\mu(x'_\rho) = \Gamma_{\mu\nu}\, A_\nu(\Gamma^{-1}_{\rho\alpha} x_\alpha)$

例: $\vec{v}(t)$, $\vec{r}(t)$,

$$\vec{v}'(t) = \vec{v}(\boxed{t'}) - \vec{V}$$

$$\vec{r}'(t') = \vec{r}(\boxed{t'}) - \vec{V}t'$$

速度方程: $\vec{v}(t) = \frac{d\vec{r}(t}{dt}$

$$\vec{v}'(t') = \frac{d\vec{r}'(t')}{dt'}$$

波动方程的伽利略变换：波速怎么变？

前面质点的速度非常清楚

$\vec{v}(t)$, $\vec{v}'(t')$

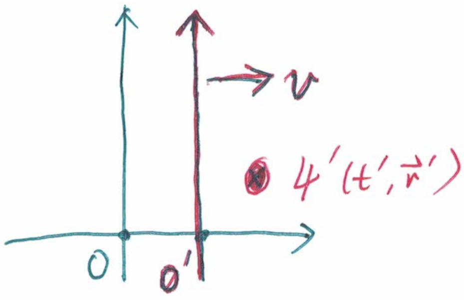
波的速度如何？

$$\psi(\vec{r}, t) = f(\omega t - \vec{k}\cdot\vec{r}),$$

满足波动方程 $\frac{\partial^2 \psi}{\partial t^2} - v^2 \nabla^2 \psi = 0$

$$\boxed{v = \frac{\omega}{k}}$$

伽利略变换：



$$\psi(t', \vec{r}') = \overset{\psi(t,\vec{r})}{f(\cancel{\omega t - \vec{k}\cdot\vec{r}})}$$

$$= f(\omega t - \vec{k}\cdot\vec{r})$$

$$\boxed{t' = t,\ \vec{r}' = \vec{r} - \vec{V}t}$$

$$= f(\omega t' - \vec{k}\cdot(\vec{r}' + \vec{V}t'))$$

$$= f(\cancel{\omega +}(\omega - \vec{k}\cdot\vec{V})t' - \vec{k}\cdot\vec{r}')$$

$$= f(\omega' t' - \vec{k}'\cdot\vec{r}') \rightarrow$$ 满足波动方程

$$\frac{\partial^2 \psi'}{\partial t'^2} - v'^2 \nabla'^2 \psi' = 0$$

$$\omega' = \omega - \vec{k}\cdot\vec{V}, \quad \vec{k} = \vec{k}'$$

$$v' = \frac{\omega}{k'} = \frac{\omega - \vec{k}\cdot\vec{V}}{k} = v - \hat{k}\cdot\vec{V}$$

~~食~~作业：直接用Lorentz变换

由此可见：波的相速度与质点满足同一个变换规则.

(贰) Maxwell 方程组与伽利略变换

介质的 Maxwell 方程组在伽利略变换下如何进行？

这是一个相对复杂的问题 （材料学家王院士）

这里讨论下真空中波动方程：

$$\frac{\partial\vec{B}}{\partial t}=-\nabla\times\vec{E}$$

$$\frac{\partial\vec{E}}{\partial t}=c^2\nabla^2\times\vec{B} \quad \Longrightarrow \quad \frac{\partial^2\vec{E}}{\partial t}-c^2\nabla^2\vec{E}=0$$

$$\nabla\cdot\vec{B}=0 \qquad\qquad \frac{\partial^2\vec{B}}{\partial t}-c^2\nabla^2\vec{B}=0.$$

$$\nabla\cdot\vec{E}=0$$

$$\boxed{c=\frac{1}{\sqrt{\varepsilon_0\mu_0}}}$$ 始终保持不变

但伽利略变换要求 $c'=c-\hat{k}\cdot\vec{v}$

这是第二个矛盾！

（第一个是 Maxwell 方程与 Lorentz 力公式的矛盾）.

→ Maxwell 方程组没有 Lorentz 变换协变性！

问题：① 光速到底是否依赖参考系

② 电场，磁场在~~各~~不同参考系下如何变换？

(3) 介质 . . .

有没有变换法则与 Maxwell 方程组"和谐"，洛伦茨给出一个例子，$\boxed{\text{现在称为 Lorentz 变换}}$

↳ Lorentz 解决了这个电动力学问题.

仅以 $\vec{V}=V_x\vec{x}$ 为例.

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta/c}{\sqrt{1-\beta^2}} & 0 & 0 \\ -\frac{\beta c}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$t' = \frac{t - Vx/c^2}{\sqrt{1-V^2/c^2}} = \frac{t-\beta x/c}{\sqrt{1-\beta^2}}$$, $\beta = \frac{V}{c}$, $V=\beta c$

$$x' = \frac{x-Vt}{\sqrt{1-V^2/c^2}} = \frac{x-\beta ct}{\sqrt{1-\beta^2}}$$

$$y' = y$$

$$z' = z$$

Lorentz 发现 $F_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & E_x & E_y & E_z \\ \hline -E_x & 0 & c^2B_z & -c^2B_y \\ -E_y & -c^2B_z & 0 & c^2B_x \\ -E_z & c^2B_y & -c^2B_x & 0 \end{array}\right)$

或者矢势 $A_\alpha = \begin{pmatrix} \varphi/c^2 \\ A_x \\ A_y \\ A_z \end{pmatrix}$ 按照4矢量规则变换，

则上述问题都得到解决。

$\boxed{\text{计算一下：}}$ 注意坐标要进行逆变换.

$$F'_{\alpha\beta} = \frac{1}{1-\beta^2} \begin{pmatrix} \cancel{\frac{1}{\sqrt{1-\beta^2}}}\,1 & \cancel{\beta/c}\,-\beta/c & 0 & 0 \\ -\beta c & 1 & 0 & 0 \\ 0 & 0 & \sqrt{1-\beta^2} & 0 \\ 0 & 0 & 0 & \sqrt{1-\beta^2} \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & c^2B_z & -c^2B_y \\ -E_y & -c^2B_z & 0 & c^2B_x \\ -E_z & c^2B_y & -c^2B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & \overset{-\beta c}{\cancel{-\beta/c}} & 0 & 0 \\ \overset{-\beta/c}{\cancel{-\beta c}} & 1 & 0 & 0 \\ 0 & 0 & \sqrt{1-\beta^2} & 0 \\ 0 & 0 & 0 & \sqrt{1-\beta^2} \end{pmatrix}$$

$$= \frac{1}{1-\beta^2} \begin{pmatrix} \beta E_x/c & E_x & E_y - \beta c B_z & E_z + \beta c B_y \\ -E_x & -\beta c E_x & -\beta c E_y + c^2 B_z & -\beta c E_z - c^2 B_y \\ -\sqrt{1-\beta^2}\,E_y & -c^2\sqrt{1-\beta^2}\,B_z & 0 & c^2\sqrt{1-\beta^2}\,B_x \\ -\sqrt{1-\beta^2}\,E_z & c^2\sqrt{1-\beta^2}\,B_y & -c^2\sqrt{1-\beta^2}\,B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & \overset{-\beta c}{\cancel{-\beta/c}} & & \\ -\beta/c & 1 & & \\ & & \sqrt{1-\beta^2} & \\ & & & \sqrt{1-\beta^2} \end{pmatrix}$$

$$= \frac{1}{1-\beta^2} \begin{pmatrix} 0 & -\beta^2 E_x + E_x & \sqrt{1-\beta^2}\,(E_y - \beta c B_z) & \sqrt{1-\beta^2}\,(E_z + \beta c B_y) \\ -E_x + \beta^2 E_x & \beta c E_x - \beta c E_x & \sqrt{1-\beta^2}\,(-\beta c E_y + c^2 B_z) & \sqrt{1-\beta^2}\,(-\beta c E_z - c^2 B_y) \\ -\sqrt{1-\beta^2}\,E_y + \beta c\sqrt{1-\beta^2}\,B_z & \beta c\sqrt{1-\beta^2}\,E_y - c^2\sqrt{1-\beta^2}\,B_z & 0 & c^2(1-\beta^2)\,B_x \\ -\sqrt{1-\beta^2}\,E_z - \beta c\sqrt{1-\beta^2}\,B_y & \beta c\sqrt{1-\beta^2}\,E_z + c^2\sqrt{1-\beta^2}\,B_y & -c^2(1-\beta^2)B_x & 0 \end{pmatrix}$$

$$= \left(\begin{array}{c|c|c|c}
0 & E_x & \frac{E_y - \beta c B_z}{\sqrt{1-\beta^2}} & \frac{E_z + \beta c B_y}{\sqrt{1-\beta^2}} \\ \hline
-E_x & 0 & \frac{-\beta c E_y + c^2 B_z}{\sqrt{1-\beta^2}} & \frac{-\beta c E_z - c^2 B_y}{\sqrt{1-\beta^2}} \\ \hline
\frac{-E_y + \beta c B_z}{\sqrt{1-\beta^2}} & \frac{\beta c E_y - c^2 B_z}{\sqrt{1-\beta^2}} & 0 & c^2 B_x \\ \hline
\frac{-E_z - \beta c B_y}{\sqrt{1-\beta^2}} & \frac{\beta c E_z + c^2 B_y}{\sqrt{1-\beta^2}} & -c^2 B_x & 0
\end{array}\right)$$

我们得到变换关系

$$E_x' = E_x \qquad\qquad B_x' = B_x$$

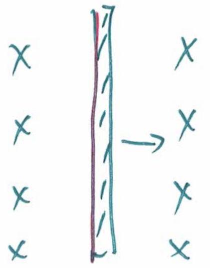
$$\boxed{E_y' = \frac{E_y - \beta c B_z}{\sqrt{1-\beta^2}}}\ \checkmark \qquad\qquad B_y' = \frac{B_y + \beta E_z / c}{\sqrt{1-\beta^2}}$$

$$\boxed{E_z' = \frac{E_z + \beta c B_y}{\sqrt{1-\beta^2}}} \qquad\qquad B_z' = \frac{B_z - \beta E_y / c}{\sqrt{1-\beta^2}}$$

$\beta = \frac{v}{c}$ ，~~$E_z' = \frac{E_z + v B_y}{\sqrt{}}$~~

$\Longrightarrow$ $\vec{E}, \vec{B}$ 并不是矢量场，而是张量场！

$$E_y' = \frac{E_y - v B_z}{\sqrt{1-\beta^2}}$$



解决问题①！

~~光速~~ 光速问题？

$$\boxed{\left(\frac{\partial}{\partial t'^2} - c^2 \nabla'^2\right) \vec{E}(t', \vec{r}\,') = 0}$$

↳ 不变量.

电磁势的变换:

$$\begin{pmatrix} \frac{\varphi'}{c^2} \\ A_x' \\ A_y' \\ A_z' \end{pmatrix} = \frac{1}{\sqrt{1-\beta^2}} \left(\begin{array}{c|c|c|c} 1 & -\frac{\beta}{c} & & \\ \hline -\beta c & 1 & & \\ \hline & & \sqrt{1-\beta^2} & \\ \hline & & & \sqrt{1-\beta^2} \end{array}\right) \begin{pmatrix} \frac{\varphi}{c^2} \\ A_x \\ A_y \\ A_z \end{pmatrix}$$

$$= \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} \frac{\varphi}{c^2} - \frac{\beta}{c} A_x \\ -\beta \frac{\varphi}{c} + A_x \\ \sqrt{1-\beta^2}\, A_y \\ \sqrt{1-\beta^2}\, A_z \end{pmatrix} \Longrightarrow$$

$$\varphi' \to \frac{\varphi - \beta c A_x}{\sqrt{1-\beta^2}}$$

$$A_x' \longrightarrow \frac{A_x - \frac{\beta}{c}\varphi}{\sqrt{1-\beta^2}}$$

$$A_y' \to A_y$$

$$A_z' \to A_z$$

光速为何不变？

$$t' = \frac{t - \beta x/c}{\sqrt{1-\beta^2}}, \quad x' = \frac{x - \beta c t}{\sqrt{1-\beta^2}}$$

$$v = \frac{\omega}{c}, \quad \boxed{x = \frac{x' + \beta c t'}{\sqrt{1-\beta^2}}, \quad t = \frac{t' + \beta x'/c}{\sqrt{1-\beta^2}}}$$

$$\psi(t', \vec{r}\,') = \psi(t, \vec{r}) = f(\omega t - \vec{k} \cdot \vec{r})$$

$$= f\left(\omega \frac{t' + \beta x'/c}{\sqrt{1-\beta^2}} - k \cdot \frac{x' + \beta c t'}{\sqrt{1-\beta^2}}\right)$$

$$= f\left(\left(\frac{\omega}{\sqrt{1-\beta^2}} - \frac{k\beta c}{\sqrt{1-\beta^2}}\right)t' - \left(-\frac{\omega\beta/c}{\sqrt{1-\beta^2}} + \frac{k}{\sqrt{1-\beta^2}}\right)x'\right)$$

$$\omega' = \frac{1}{\sqrt{1-\beta^2}}(\omega - k\beta c) = \frac{c\beta}{\sqrt{1-\beta^2}}\left(\frac{\omega}{\beta c} - k\right)$$

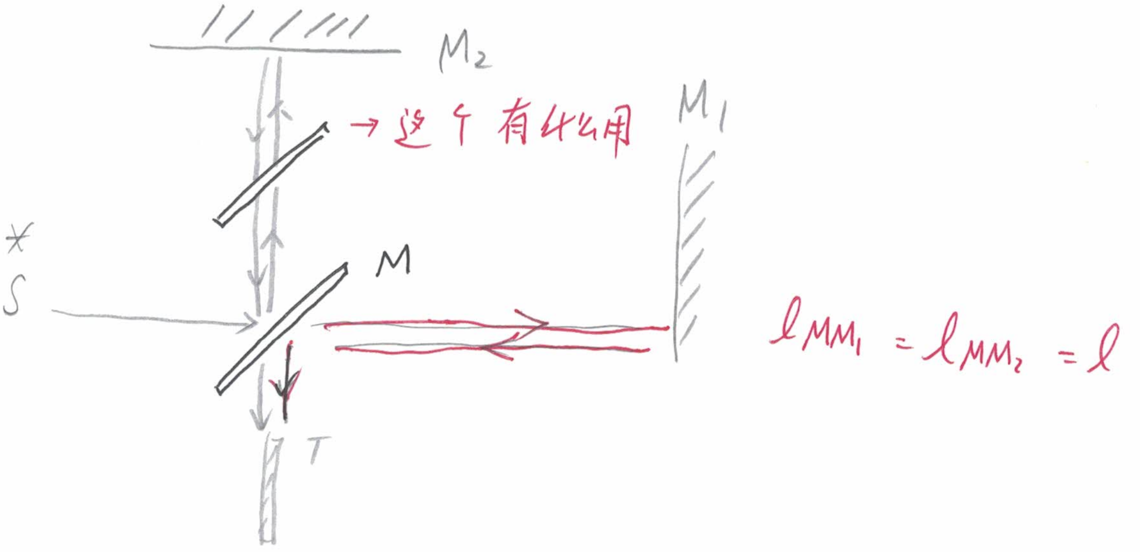
$$k' = \frac{1}{\sqrt{1-\beta^2}}\left(k - \frac{\omega\beta}{c}\right)$$

$$\frac{\omega'}{k'} = c\beta\,\frac{\frac{\omega}{\beta c} - k}{k - \frac{\omega\beta}{c}} = c\beta\,\frac{\frac{k}{\beta} - k}{k - \beta k}$$

$$= c\beta\,\frac{\frac{1}{\beta} - 1}{1-\beta} = c$$ 光速保持不变

实验基础：

Michelson - Morley（迈克耳孙 - 莫雷）实验



Lorentz 变换的性质:

⓪. $$x' = \frac{x-\beta ct}{\sqrt{1-\beta^2}}$$

$$t' = \frac{t-\beta x/c}{\sqrt{1-\beta^2}}$$

①. "同时"的相对性.

$$\Delta t' = \frac{\Delta t - \beta\frac{\Delta x}{c}}{\sqrt{1-\beta^2}}$$

$$\Delta t = 0, \quad \Delta x \neq 0, \quad \Delta t' = -\frac{\beta/c}{\sqrt{1-\beta^2}}\Delta x$$

若两个不同位置发生的事件, 在 K系中同时, 则在 K'系中不同时!

②. 空间距离的相对性

$$\Delta x' = \frac{\Delta x - \beta c\Delta t}{\sqrt{1-\beta^2}} \Longrightarrow \boxed{\Delta x' = \sqrt{1-\beta^2}\,\Delta x}$$

若 $$\Delta t' = \frac{\Delta t - \beta\frac{\Delta x}{c}}{\sqrt{1-\beta^2}} = 0$$

$$\hookrightarrow \quad \Delta t = \beta\frac{\Delta x}{c} \Longrightarrow \Delta x' = \frac{\Delta x - \beta^2\Delta x}{\sqrt{1-\beta^2}} \Uparrow$$

在K系中 Δx 长的尺子，在K'系中同时测量尺子两端坐标得到的尺子长度变短！

③. 时间间隔是相对的

$$\Delta x' = \frac{\Delta x - \beta c\Delta t}{\sqrt{1-\beta^2}} = 0 \Longrightarrow \Delta x = \beta c\Delta t$$

$$\Delta t' = \frac{\Delta t - \beta\frac{\Delta x}{c}}{\sqrt{1-\beta^2}} = \cancel{\beta\Delta t}\ \sqrt{1-\beta^2}\ \Delta t$$

若K系中一个人驾车行驶了 Δx 距离，耗时 Δt.

则在与车固定的K'系中，驾驶时间变短 $\Longrightarrow$ 驾车越多越年轻

④. 不变量

$$(\Delta s)^2 = c^2(\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]$$

不变.

$$(\Delta t', \Delta x', \Delta y', \Delta z')\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = ?$$

$$\boxed{\Gamma_{\mu\alpha}\, \eta_{\alpha\beta}\, \Gamma_{\beta\nu} = \eta_{\mu\nu}}$$

证明.